**IMMUNIZATION OF BOND PORTFOLIOS:
SOME NEW RESULTS**

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Olivier de La Grandville

DISCUSSION PAPER 01.26

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Olivier de La Grandville¹

All losses are restor'd, and sorrows end

Sonnets

Thou quiet soul, sleep thou a quiet sleep

Dream of success and happy victory!

Richard III

Individual and collective security are among the gauges of civilization, and at the same time they constitute necessary conditions for its progress. Their systematic pursuit is relatively recent among nations: prompted by the dreadful lessons of the 20th century, they took their present shape after World War II. Only at that time serious consideration started being given to social and economic security, which resulted into insurance and pension plans being widespread. Large savings, concomitant to decades of economic progress, were invested into default-free vehicle, by those institutions which were looking for steady, fixed, income flows.

In the world of fixed exchange rates which followed the Bretton-Woods agreements, it was only natural that interest rates remained relatively stable.

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When that state of affairs was disrupted some three decades later, volatile interest rates transformed so-called fixed income assets into risky investments. It is at that time that considerable research was undertaken to protect institutional investors against the fluctuations of interest rates and of bond prices. The word “immunization” describes the steps taken by a bond manager to build up a portfolio which will be minimally sensitive to interest rate shifts. This study has two aims: to give a comprehensive presentation of the immunization problem; to provide novel, original results.

Let us first explain briefly what is at stake. What are the exact consequences of a rise, for instance, in interest rates? For a bond holder, is it good or bad news? The answer to this question depends crucially on the *horizon* of the investor. It is easy to figure out that an investor with a short horizon will suffer from a rise in interest rates because her bond (or bond portfolio) will immediately decrease in value, and this capital loss may not be compensated by the reinvestment of coupons at a higher rate. On the other hand, an investor with a longer horizon may welcome such a rise in interest rates for the two following reasons: first, the initial capital loss will start vanishing as bond prices return to their par value, and second, coupons will be reinvested at a higher rate. We may then surmise that under special circumstances there may exist an intermediate horizon for which the investor may be indifferent to a change in interest rates because capital gains (or losses) are nearly exactly compensated by losses (or gains) in coupon reinvestment.

Such a horizon exists, indeed; it is equal to the *duration* of the portfolio's bond, which is the weighted average of the portfolio's times of payment, the weights being the shares of the portfolio's present value cash flows in the portfolio's total value. This prompts to ask the natural question: reversing the problem, consider an investor's horizon as *given*. Is it possible to constitute a bond portfolio such that the investor is protected (immunized) against *any* change in the interest rate structure? The problem was well understood and tackled a few decades ago under simple hypotheses: the interest rate structure (the interest rates corresponding to various maturities) was supposed to receive parallel shifts. The recipe was basically to build a portfolio such that its duration was equal to the investor's horizon. In order to take into account the fact that short rates were usually more volatile than long ones, it was later proposed to equate not only the first moment (the duration) of the portfolio to the horizon, but some of its successive moments to the successive integer powers of the horizon (see D. Chambers and S. Nawalkha, eds. (1999)) – a portfolio's moment of order k is the natural generalization

if the concept of duration: it is the weighted average of the k th power of its times of payment.

No analytic foundation, however, seems to have been given to the question of determining how many moments should be taken into consideration. Indeed, it seems that no link was established between first- and second order conditions for immunization, on the one hand, and the number of moments to be considered. It is even difficult to figure out how second order conditions for a minimum of the portfolio's future value have been brought into the picture, if at all. Finally, present theory is silent on the question of the minimum number of bonds to include in any immunizing portfolio. This study intends to fill in those gaps. To do so, we will propose a general method which will always lead to a local minimum of the portfolio's future value.

Our first task will be to lay out carefully our tools of analysis: the instantaneous forward rate and the continuously compounded spot rate (Section 1). The reason for paying particular attention to those tools is that, in our opinion, they are not well understood, as the following examples illustrate. First consider the well-known, usual, shape of the spot rate curve (the spot rate as a function of maturity): increasing, concave, leveling off to a plateau. Neither practitioners nor academics seem to realize that the corresponding forward curve *must be decreasing* before the spot rate reaches its plateau. This failure is evidenced by the various descriptions of the companion forward curve: the fruit of remarkable imagination, they unfortunately never exhibit the right properties. On a more general level – and this may explain the shortcomings just mentioned – it is not recognized that the forward instantaneous rate and the continuously compounded spot rate play *exactly* the same roles as a marginal and an average quantity in economics. The reason for this poor perception may be the following: while marginal and average values in economics are derived from well-known quantities (for example, total cost, or total revenue), the common variable giving rise here to marginal and average values is well-hidden: to the best of our knowledge, it has not yet even been defined. We will need to introduce it, and we will call it the “continuously compounded *total* return”.

In Section 2, we will present the concept of immunization for those readers who are not already familiar with it, in the following, simple framework. We will consider that the spot rate curve undergoes parallel shifts. We will then show that under those special circumstances duration plays indeed a central role in the immunization process, in the sense that a portfolio is completely immunized if its duration is equal to the investor's horizon.

Section 3 will be devoted to stating and proving a general immunization theorem. In order to cope with real world cases, where the spot rate curve can receive any kind of shift, it will be necessary to recall the concept of the k th moment of a bond portfolio. Its properties are essential, and will be carefully spelt out. The key to successful immunization will be to build a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive definite. The way to achieve this last aim will be to equate one of the elements of the Hessian matrix to a positive, arbitrary, parameter. We will examine the important consequences of these conditions in terms of the total number of bonds to be considered in the immunizing portfolio, and of the structure of the portfolio. Finally, the theorem will be put to the test in Section 4, where numerous applications will be given. We will consider completely arbitrary initial spot rate curves, as well as large, arbitrary shifts of those.

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1 Basic concepts for valuation and immunization of bond portfolios in continuous time

1.1 The instantaneous forward rate

Consider a time span $[u, v]$ subdivided into n intervals $\Delta z_1, \Delta z_2, \dots, \Delta z_j, \dots, \Delta z_n$. This partitioning of $[u, v]$ is arbitrary in the following sense: each Δz_j interval ($j = 1, \dots, n$) has arbitrary length; two given intervals, say Δz_1 and Δz_2 , may or may not be equal. Such an arbitrary partitioning of the time span $[u, v]$ is represented in Figure 1.

Fig. 1. An arbitrary partitioning of the time span $[u, v]$

Suppose that to each interval Δz_j corresponds a forward rate. Such a forward rate is defined as the *yearly* interest rate for a loan agreed upon at time u ; the loan is to start at the beginning of the Δz_j interval, and is to be paid back at the end of Δz_j . Call f_j this forward rate. For instance, f_j may be equal to 5% per year.

If \$1 is lent at the beginning of this time interval Δz_j , what will be the amount to be reimbursed at the end of Δz_j ? There are two ways of answering this question. One could agree that it is:

- either $(1 + f_j)^{\Delta z_j}$, in conformity with annual interest compounding if Δz_j is typically equal to a few years
- or $1 + f_j \Delta z_j$, following what is usually done if the interval Δz_j is smaller than one year, which will be justified below.

Until now, we have said nothing of the length of the time span $[u, v]$. In fact, inasmuch as the partitioning of $[u, v]$ is arbitrary, so is its length: it could be a non integer number of years or days. It turns out that, whatever this length may be, we will at one point consider that the number n of intervals Δz_j tends towards infinity, and that at the same time the largest of those intervals tends towards zero. Therefore, we can assume already at this stage that Δz_j is small. We will now show why it is both natural and legitimate to use the second formula, i.e. $1 + f_j \Delta z_j$.

that Δz_j is small. We will now show why it is both natural and legitimate to use the second formula, i.e. $1 + f_j \Delta z_j$.

This formula is an excellent, *second-order*, approximation of the first formula $(1 + f_j)^{\Delta z_j}$. Indeed, consider the latter expression as a function of the two variables f_j and Δz_j , which we will for convenience denote momentarily as f and h respectively. Let $\varphi(\cdot)$ be the two-variable function

$$\varphi(f, h) = (1 + f)^h \equiv (1 + f_j)^{\Delta z_j}. \quad (1)$$

Develop (1) in Taylor series around point $(f, h) = (0, 0)$. The second-order approximation is:

$$\begin{aligned} \varphi(f, h) \approx & \varphi(0, 0) + \frac{\partial \varphi}{\partial f}(0, 0)f + \frac{\partial \varphi}{\partial h}(0, 0)h \\ & + \frac{1}{2} \left[\frac{\partial^2 \varphi}{\partial f^2}(0, 0)f^2 + 2 \frac{\partial^2 \varphi}{\partial f \partial h}(0, 0)fh + \frac{\partial^2 \varphi}{\partial h^2}(0, 0)h^2 \right]. \end{aligned} \quad (2)$$

Immediate calculations show that among the six terms in the right-hand side of (2), all are equal to zero except the first one, equal to 1, and the fifth (involving the cross second-order derivative), equal to fh . Therefore, $\varphi(f, h) \approx 1 + fh$ and we may write

$$(1 + f_j)^{\Delta z_j} \approx 1 + f_j \Delta z_j \quad (3)$$

as a second-order approximation. To get some sense of the closeness of this approximation, suppose that f_j is 5% per year and Δz_j is one month, or 0.083 year. On the one hand, $(1 + f_j)^{\Delta z_j} = 1.0041$, while $1 + f_j \Delta z_j = 1.0042$ on the other.

Consider now that the rate of interest f_j is compounded not once within interval Δz_j , but m times. Using this approximation, one dollar invested at the beginning of this interval thus becomes $(1 + f_j \Delta z_j / m)^m$ where $\Delta z_j / m$ is the length of time over which the interest is compounded. Now set $f_j \Delta z_j / m = 1/k$; thus $m = k f_j \Delta z_j$ and one dollar becomes $(1 + 1/k)^{k f_j \Delta z_j}$. Take the limit of this amount when the number of compounding m within Δz_j tends towards infinity – equivalently, when $k \rightarrow \infty$, since the $f_j \Delta z_j$'s terms are finite. We get:

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^{k f_j \Delta z_j} = e^{f_j \Delta z_j}. \quad (4)$$

Note that this result could have been reached without recourse to the second-order approximation $(1+f_j)^{\Delta z_j} \approx 1+f_j\Delta z_j$. From formula $(1+f_j)^{\Delta z_j}$, when interest is compounded m times within Δz_j , we get $(1+f_j/m)^{m\Delta z_j}$ at the end of interval Δz_j . Now set $f_j/m = 1/k$, so that $m = kf_j$. One dollar becomes $(1+1/k)^{kf_j\Delta z_j}$, and the above result obtains.

It is clear that if the same limiting process is taken for each interval Δz_j ($j = 1, \dots, n$), an investment C_u at time u becomes

$$C_v = C_u e^{\sum_{j=1}^n f_j \Delta z_j} \quad (5)$$

at time v .

Suppose now that, whatever the initial partitioning of the time span $[u, v]$, the sum $\sum_{j=1}^n f_j \Delta z_j$ tends toward a unique limit when the number of time intervals n tends toward infinity, and when the maximum interval length Δz_j tends toward zero. The forward rate function is then said to be integrable, and the limit is the definite integral of f_j over $[u, v]$. We have:

$$\lim_{\substack{n \rightarrow \infty \\ \max \Delta z_j \rightarrow 0}} \sum_{j=1}^n f_j \Delta z_j = \int_u^v f(z) dz. \quad (6)$$

We are now ready for a definition of the instantaneous forward rate.

• **Definition.** Let $[u, v]$ be a time span. Let z be any point of time within this interval. The *instantaneous forward rate* $f(z)$ is the yearly interest rate decided upon at time u for a loan starting at time z , $z \in [u, v]$, for an infinitesimally small trading period dz .

Sometimes, to recall the time (u) at which this instantaneous forward rate is agreed upon, the forward rate is denoted $f(u, z)$. If, for instance, the time of inception of the contract is 0 and the time at which the loan starts is T (with an infinitesimal trading period), the instantaneous forward rate is denoted $f(0, T)$, and so forth.

From (5) and (6) a sum C_u invested during $[u, v]$ at the infinite number of instantaneous forward rates $f(u, z)$ – or $f(z)$ for short – becomes

$$C_v = C_u e^{\int_u^v f(z) dz}. \quad (7)$$

A final word about (6) and (7) is in order. Notice the dimension of the infinitesimal element $f(z)dz$ in the integral: since $f(z)$ is in $(1/\text{time})$ units, and since dz is in time units, $f(z)dz$ is dimensionless; and so are $\int_u^v f(z)dz$ and $\exp[\int_u^v f(z)dz]$.

1.2 The continuously compounded spot rate

Consider the following, particular, rate of interest:

- the time of signing of the contract and the starting point of the loan are at the same point of time: u .
- the length of the loan is the time span $[u, v]$.
- the amount loaned is C_u ; the amount due is C_v .

There are a number of ways of defining the rate of interest on such a loan. If this rate is calculated once only over the whole period $[v - u]$ (which may or may not be an integer number of years), one could define this rate as the *spot* rate (because it corresponds to a loan starting at the very signing of the contract) compounded *once* over $[v - u]$. Denote such a spot rate as $s_{u,v}^{(1)}$. The lower indexes of s are self-explanatory; the superscript (1) refers to the fact that the rate of interest is calculated once over $[u, v]$. We have

$$s_{u,v}^{(1)} = \frac{C_v - C_u}{C_u} / (v - u) \quad (8)$$

and equivalently

$$C_v = C_u[1 + s_{u,v}^{(1)}(v - u)]. \quad (9)$$

Suppose now that our spot interest is compounded m times over interval $(v - u)$ instead of once only. It implies that the length of time between any two successive compoundings is $(v - u)/m$. The spot rate is now written $s_{u,v}^{(m)}$, and equation (9) becomes

$$C_v = C_u[1 + s_{u,v}^{(m)}(v - u)/m]^m. \quad (10)$$

In order for the contracts to be equivalent (that is, to yield the same amount due C_v), the right-hand sides of (9) and (10) must be the same. We must have

$$1 + s_{u,v}^{(1)}(v - u) = [1 + s_{u,v}^{(m)}(v - u)/m]^m \quad (11)$$

or

$$s_{u,v}^m = \frac{m}{v-u} \left\{ \left[1 + s_{u,v}^{(1)}(v-u) \right]^{1/m} - 1 \right\}. \quad (12)$$

Note that replacing m by 1 in the right-hand side of (12) yields $s_{u,v}^{(1)}$ as it should.

Consider now what happens to (10) when the number of compoundings m tends toward infinity. In the same vein as what we did before (§1.1), we can replace $s_{u,v}^{(m)}(v-u)/m$ by $1/k$; therefore, with $m = k s_{u,v}^{(m)}(v-u)$, we can write:

$$C_v = C_u \left(1 + \frac{1}{k} \right)^{k s_{u,v}^{(m)}(v-u)}. \quad (13)$$

Taking the limit of C_v when $m \rightarrow \infty$ and $k \rightarrow \infty$:

$$\lim_{\substack{m \rightarrow \infty \\ k \rightarrow \infty}} C_v = C_u e^{s_{u,v}^{(\infty)}(v-u)}. \quad (14)$$

This leads to the following definition.

• **Definition.** Let $[u, v]$ be a time span. The *continuously compounded spot rate* is the yearly interest rate, denoted $s(u, v)$ that transforms an investment C_u at time u into C_v at time v , the rate being compounded over infinitesimally small time intervals between u and v .

Denote for simplicity $s_{u,v}^{(\infty)} \equiv s(u, v)$. Equation (14) can be written as

$$C_v = C_u e^{s(u,v)(v-u)}. \quad (15)$$

From (15), we deduce

$$s(u, v) = \frac{\ln(C_v/C_u)}{v-u}. \quad (16)$$

There is an all-important relationship between the continuously compounded spot rate $s(u, v)$ (a number) and the forward rate $f(z)$ (a function defined over the interval $[u, v]$). Arbitrage without transaction costs guarantees that the right-hand sides of (7) and (15) must be identical. Thus, from

$$e^{\int_u^v f(z)dz} = e^{s(u,v)(v-u)}$$

we get

$$s(u, v) = \frac{\int_u^v f(z) dz}{v - u}. \quad (17)$$

Therefore the spot rate $s(u, v)$ is nothing else than the average value of the forward rate function $f(z)$ over the interval $[u, v]$. Notice that in this average the infinitely large number of elements $f(z)$ are all weighted by the infinitely small quantities $dz/(v - u)$ whose sum is equal to one.

On the other hand, it is always possible to express a forward rate as a function of a spot rate. Write (17) as

$$\int_u^v f(z) dz = (v - u) s(u, v) \quad (18)$$

and take the derivative of (18) with respect to v . This yields

$$f(v) = s(u, v) + (v - u) \frac{ds(u, v)}{dv}.$$

We should stress that the instantaneous forward rate and the continuously compounded spot rate play respectively exactly the roles of a marginal and an average quantity, in economics parlance. This fact does not seem to be known, and for that reason we will now introduce what we feel to be the missing link between the two concepts, namely the continuously compounded *total* return.

1.3 Introducing the missing link: the continuously compounded total return

Let us first simplify our notation. Without loss of generality, call 0 instant u , and T instant v . Our time span $[u, v]$ is now replaced by $[0, T]$. The simplification comes from the fact that T stands both for an *instant* of time and for the *length* of the time span $[0, T]$. With this notation, our spot rate compounded once over period T , corresponding to equation (8), is now written

$$s_{0,T}^{(1)} = \frac{C_T - C_0}{C_0} / T \quad (19)$$

implying

$$s_{0,T}^{(1)}T = \frac{C_T - C_0}{C_0}.$$

This last quantity can be called the total return compounded once over period T . Contrary to the forward rate and the spot rate, which were expressed in (1/time) units, this magnitude is unitless. (Notice also that to underline this fact, we have not called $s_{0,T}^{(1)}T$ the total *rate* of return, but the total return). Equation (19) implies

$$C_T = C_0 \left[1 + s_{0,T}^{(1)}T \right]. \quad (20)$$

Suppose now that this total return corresponds to a spot rate compounded n times over the same interval T . It is thus defined as the product of that spot rate and the time length T . Call $s_{0,T}^{(n)}$ the corresponding rate of return per year. The length of time after which this rate is compounded is T/n . Therefore equation (20) is transformed into

$$C_T = C_0 \left(1 + s_{0,T}^{(n)} \frac{T}{n} \right)^n = C_0 \left(1 + \frac{s_{0,T}^{(n)}}{n} T \right)^n. \quad (21)$$

The total return when the spot rate is compounded n times over T is denoted as $s_{0,T}^{(n)}T$, and remains *dimensionless*.

From (21) and (20), we have

$$s_{0,T}^{(n)}T = n \left[(C_T/C_0)^{1/n} - 1 \right] = n \left[(1 + s_{0,T}^{(1)}T)^{1/n} - 1 \right]. \quad (22)$$

Setting $n = 1$ into (22), we verify that we get (19) as we should.

Let $n \rightarrow \infty$. We will now determine the limit of the corresponding, unitless, total return $s_{0,T}^{(\infty)}T$. We have to calculate

$$s_{0,T}^{(\infty)}T = \lim_{n \rightarrow \infty} s_{0,T}^{(n)}T = \lim_{n \rightarrow \infty} n \left[(C_T/C_0)^{1/n} - 1 \right]. \quad (23)$$

which will be called the *continuously compounded total return*.

Set $n = 1/\alpha$; then

$$s_{0,T}^{(\infty)}T = \lim_{n \rightarrow \infty} s_{0,T}^{(n)}T = \lim_{\alpha \rightarrow 0} \{ [(C_T/C_0)^\alpha - 1] / \alpha \}. \quad (24)$$

Applying L'Hospital's rule to (24), we get:

$$s_{0,T}^{(\infty)}T = \lim_{\alpha \rightarrow 0} [(C_T/C_0)^\alpha \ln(C_T/C_0)] = \ln(C_T/C_0). \quad (25)$$

We had previously denoted $s_{0,T}^{(\infty)} \equiv s(0,T)$. So the continuously compounded total return $s(0,T)T$ will be denoted $S(0,T)$ for short. We thus have:

$$S(0,T) = \ln(C_T/C_0).$$

Let now $f(0,z)$ be the forward rate decided upon at time 0, for an infinitesimal trading period starting at time z , with z between 0 and T . In the case of an infinite number of forward markets and in the absence of transaction costs, we know that arbitrage will enforce the equalities:

$$C_T = C_0 e^{s(0,T)T} = C_0 e^{\int_0^T f(0,z)dz}$$

from which $S(0,T) = \log(C_T/C_0) = s(0,T)T = \int_0^T f(0,z)dz$ results. Notice that the continuously compounded total return is dimensionless, as it should.

We may now state a formal definition of the continuously compounded total return.

1.3.1 Definition of the continuously compounded total return

Let $f(0,z)$ be an instantaneous forward rate agreed upon at 0 for a loan starting at z ($0 \leq z < T$), for an infinitesimal trading period. We define the *continuously compounded total return*² at time T as the integral sum of the forward rates between 0 and T , equal to $\int_0^T f(0,z)dz$. It is denoted $S(0,T)$, and is a pure (dimensionless) number.

From what precedes, we can state the following properties:

- Property 1. Let $s(0,T)$ be a continuously compounded spot rate for a loan started at time 0 and maturing at T . Then the total return $S(0,T)$ is equal to $s(0,T)T$.
- Property 2. Let C_0 denote the value of an investment at time 0; C_T is its value at time T . Then the total return $S(0,T)$ equals $\ln(C_T/C_0)$.

²Further abbreviated as the *total return*. From now on, the continuously compounded spot rate $s(0,T)$ will be abbreviated as "spot rate", and "forward rate" will stand for the instantaneous forward rate $f(0,T)$.

An example may be useful. Suppose the total return is 32% or 0.32 after four years. It means that one dollar becomes $e^{0.32} = 1.377$ dollars after four years. Equivalently, it also means also that

- the continuously compounded yearly return, or the spot rate with 4-years maturity is $0.32/4 = 0.08$ per year.
- the forward rate function for maturities between $t = 0$ and $t = 4$ is *any* function $f(z)$ such that

$$\int_0^4 f(z)dz = 0.32,$$

or that its average value is $\int_0^4 f(z)dz/4 = 0.08$ per year. In order to get such a function $f(0, z)$, choose *any* differentiable function $s(0, z)$ with $z \in [0, 4]$ that goes through point $(4, 0.08)$. Then $f(0, z)$ is simply $s(0, z) + z ds(0, z)/dz$.

- the yearly, once-a-year compounded spot rate, $s_{0,4}$, is such that

$$(1 + s_{0,4})^4 = e^{0.32}$$

or

$$s_{0,4} = e^{0.08} - 1 = 0.083 \text{ per year}$$

Those definitions of the forward rate, the spot rate and the total return enable us to write the following equalities:

$$C_T = C_0 e^{\int_0^T f(z)dz} = C_0 e^{s(0,T)T} = C_0 e^{S(0,T)}. \quad (26)$$

Noticing that the first equality in (26) can be written as

$$C_T = C_0 e^{[\frac{1}{T} \int_0^T f(z)dz]T}$$

equalities (26) allow for a nice interpretation of the Euler number e . The Euler number, e , is equal to what becomes of \$1 when, equivalently:

- the *average* of the forward rates over period T is $1/T$
- the spot rate $s(0, T)$ is equal to $1/T$
- the continuously compounded total return is 1 (100%).

As an example, suppose that $T = 25$ years. Then $1/T$ is 4% per year. One dollar becomes $e = 2.718...$ dollars if the *average* of the forward rates over 25 years – equivalently the spot rate $s(0, 25)$ – is 4%. Notice that this average of 4% allows for *negative* values of the forward rate inasmuch as the

forward rate is a *real* forward rate, defined as the difference between the nominal forward rate and the inflation rate.

We will now examine the precise relationships existing between the three concepts we have defined. Understanding these relationships seems essential to avoid innumerable erroneous depictions of forward and spot rate curves such as those appearing regularly in otherwise excellent texts on finance. Indeed, all too often the traditional increasing, concave spot curve leveling off to a plateau is accompanied by a forward curve which has simply no relevance to it. The latter's position can be way off the mark, being faulty on many grounds. In particular, before the spot rate reaches a plateau, it is depicted as still increasing while it must be *decreasing* – this is a property that will be demonstrated in §1.5. Also, it turns out that texts are sometimes graced with hump-shaped spot rate curves; not infrequently the decreasing part of the hump exhibits a subtangent (measured on the ordinate) *larger* than the spot rate³, thus entailing nonsensical *negative* nominal forward rates.

1.4 Relationships between the total return, the forward rate and the spot rate

It is now straightforward to show that the forward rate and the spot rate are, respectively, the marginal and average quantities of a common value, the total return.

First, consider the total return as expressed as an integral sum, and take its derivative with respect to T . We get the forward rate:

$$\frac{dS(0, T)}{dT} = \frac{d}{dT} \left[\int_0^T f(0, z) dz \right] = f(0, T). \quad (27)$$

In economic parlance, the forward rate is simply the marginal return.

³The subtangent of $s(0, T)$ is defined as the absolute value of the differential $T ds(0, T)/dT$. If the spot rate is decreasing, we must still have, with $ds(0, T)(dT < 0)$:

$$f(T) = s(0, T) + T \frac{ds(0, T)}{dT} > 0$$

or

$$-T \frac{ds(0, T)}{dT} = \left| T \frac{ds(0, T)}{dT} \right| < s(0, T)$$

Second, take the average value of $S(0, T)$. We get the spot rate

$$\frac{S(0, T)}{T} = \frac{\int_0^T f(0, z) dz}{T} = \frac{s(0, T)T}{T} = s(0, T). \quad (28)$$

Thus the spot rate is the average return.

Taking the derivative of $S(0, T) = s(0, T)T$ with respect to T and using (27) we get

$$f(0, T) = s(0, T) + \frac{ds(0, T)}{dT}T. \quad (29)$$

This shows that the forward rate can be constructed geometrically by adding to the spot rate $s(0, T)$ the differential $[ds(0, T)/dT]T$ (see §1.6).

In order to avoid some of the errors we alluded to at the end of §1.3, it is important to understand the full meaning of the differential $[ds(0, T)/dT]T$ of the spot curve. If $s(0, T)$, the average value of the forward rates, is an increasing function of T , the forward rate (the marginal value of the *total return*) is equal to the average value (the spot rate) plus the increase in the average that now applies to *all* infinitesimal time increments from 0 to T . This increase is $[ds(0, T)/dT]T$; indeed it is the rate of increase per additional time to maturity $[ds(0, T)/dT]$ multiplied by the sum of all increases in maturity from 0 to T , i.e. $\int_0^T dz = T$. Thus, the geometrical construction of the forward rate from the spot rate as $f(0, T) = s(0, T) + [ds(0, T)/dT]T$ makes sound economic sense. As an application, we can understand immediately what is at stake when representing a spot rate curve with a hump: the decreasing part of the hump may be too steep, in the sense that $[ds(0, T)/dT]T < -s(0, T)$ over some interval of T , thus entailing $s(0, T) + [ds(0, T)/dT]T < 0$ and hence non-sensical negative (nominal) forward rates $f(0, T)$ over that interval.

Figure 2 summarizes the relationships between the three concepts of total return, spot rate and forward rate. As the reader can verify, there is a one-to-one correspondence between these three concepts, and always two ways of going from one of these to any of the other two: a direct one and an indirect one. These relationships will be put to work on an equal basis in the next two sections.

Figure 2. Summary of the relationships between the total return $S(0, T)$, the spot rate $s(0, T)$ and the forward rate $f(0, T)$

1.5 Theorems on the behavior of the forward rate and the total return

We will now state and demonstrate a property of the forward rate and the total return that pertains to the most commonly observed behavior of the spot rate.

1.5.1 Theorem 1

Let the spot rate $s(0, T)$ be represented by an increasing, concave function of maturity T , leveling off at $s(0, \bar{T}) = \bar{s}$ for an abscissa $\bar{T}$, and equal to level $\bar{s}$ beyond $\bar{T}$. Then:

a) the forward rate goes at least through one maximum in the interval $(0, \bar{T})$ and is decreasing in an interval to the left of $\bar{T}$. If $T \geq \bar{T}$, the forward rate is equal to the spot rate $\bar{s}$.

b) the total return is an increasing function of T , with at least one inflection point for $0 < T < \bar{T}$. If $T \geq \bar{T}$, the total return becomes a linear function of T .

1.5.2 Proof

Let us first consider the behavior of the slope of the forward rate curve at the origin, i.e. when $T \rightarrow 0$. The above assumptions imply $ds(0, 0)/dT > 0$. On the other hand, taking the derivative of (34) with respect to T , we deduce

$$\frac{df(0, T)}{dT} = \frac{ds(0, T)}{dT} + \frac{d^2s(0, T)}{dT^2}T + \frac{ds(0, T)}{dT} = 2\frac{ds(0, T)}{dT} + \frac{d^2s(0, T)}{dT^2}T \quad (30)$$

and therefore, with $T = 0$

$$\frac{df(0, 0)}{dT} = 2\frac{ds(0, 0)}{dT}. \quad (31)$$

(Notice how surprisingly simple this intermediate result is: *whatever* the spot rate curve, and in particular *whatever the sign* of the spot curve slope at the origin, the initial slope of the forward rate curve will *always* be twice that of the spot curve⁴).

⁴This result can be generalized while retaining its simplicity. It is easy to show that at the origin the n^{th} derivative of the forward rate is $(n + 1)$ times the n^{th} derivative of the spot rate, i.e. $d^n f(0, 0)/dT^n = (n + 1)d^n s(0, 0)/dT^n$.

Coming back to (30), we know that for $T = \bar{T}$, $ds(0, \bar{T})/dT = 0$ and its left-hand side second derivative is negative ($d^2s(0, \bar{T}_-)/dT^2 < 0$), as implied in our concavity hypothesis. This leads to a left-hand side derivative of $f(0, \bar{T})$ equal to

$$\frac{df(0, \bar{T}_-)}{dT} = \frac{d^2s(0, \bar{T}_-)}{dT^2} \bar{T} < 0$$

and therefore the forward rate is *decreasing* in an interval on the left-hand side of $\bar{T}$.

We know that $df(0, 0)/dT$ is strictly positive and that $df(0, \bar{T}_-)/dT$ is strictly negative. It follows that $df(0, T)/dT$ changes sign from positive to negative at least once within the interval $[0, T]$, and therefore that $f(0, T)$ goes through a local maximum at least for one value of the maturity. We denote such a value $\hat{T}$ ($\hat{T} \in (0, \bar{T})$) – assuming it is unique.

The final part of part a) of the theorem is evident from (29): if $ds(0, T)/dT = 0$ for $T > \bar{T}$, then $f(0, T) = s(0, T)$.

Part b) is just a direct consequence of a). If $f(0, T)$ goes through a unique maximum in the interval $[0, T]$ (at $T = \hat{T}$), its integral (the total return $S(0, T)$) has an inflection point at $\hat{T}$; it is a convex function between 0 and $\hat{T}$ and a concave function between $\hat{T}$ and $\bar{T}$. Finally, if for $T \geq \bar{T}$ $f(0, T)$ is equal to a constant $s(0, T) \equiv \bar{s}$, then

$$\begin{aligned} S(0, T)|_{T \geq \bar{T}} &= \int_0^T f(0, z) dz \Big|_{T \geq \bar{T}} \\ &= \int_0^{\bar{T}} f(0, z) dz + \int_{\bar{T}}^T f(0, z) dz \Big|_{T \geq \bar{T}} \\ &= \bar{s}\bar{T} + \bar{s}T - \bar{s}\bar{T} \Big|_{T \geq \bar{T}} = \bar{s}T \Big|_{T \geq \bar{T}}, \end{aligned}$$

a *linear* function of maturity T . The proof is thus complete⁵. Note that this theorem could be generalized in a natural way to the relationship between the derivative of a function and the average value of that function.

⁵This theorem allows for the following interpretation. Consider the continuous maturity variable as a discrete one, albeit with infinitesimally short increments. First, let us explain the property that at the origin the forward rate is equal to the spot rate, and that the forward curve has twice the slope of the spot curve. Let ε ($\varepsilon \rightarrow 0$) be the first maturity, 2ε the second one. For only one maturity, the spot rate is the average of one forward rate only; therefore it is equal to it. So $s(0, 1) = f(0, 0)$. Suppose now that the spot rate for maturity 2ε has increased by an amount Δs . If an average increases by Δs for one

A corollary to this theorem is the following:

1.5.3 Theorem 2

Let the spot rate be represented by a decreasing, convex function of maturity T , leveling off at $s(0, \bar{T}) = \bar{s}$ for an abscissa $\bar{T}$, and equal to level $\bar{s}$ beyond $\bar{T}$. Then:

a) the forward rate goes at least through one minimum in the interval $(0, T)$ and is increasing in an interval to the left of $\bar{T}$. If $T \geq \bar{T}$, the forward rate is equal to the spot rate $\bar{s}$.

b) the total return is an increasing function of T , with at least one inflection point for $0 < T < \bar{T}$. If $T \geq \bar{T}$, the total return becomes a linear function of T .

1.5.4 Proof

The proof follows the same lines as above.

1.6 The spot rate curve as a spline and its corresponding forward rate curve

In order to derive a correct forward curve corresponding to a spot rate curve, we could proceed geometrically and obtain a fair result by adding to the spot rate its differential $T \cdot ds(0, T)/dT$. Consider any value of the maturity, for

additional element, it implies that the second element has increased by twice the increase of the average, i.e. by $2\Delta s$. (The same argument carries of course to the case where the average *decreases* by Δs). Therefore at the origin the slope of the forward rate must be twice the slope of the spot rate. Furthermore, at the point where the spot rate (the average of the forward rates) reaches a maximum, the last forward rate must be equal to the spot rate. The fact that between the origin and that point the forward rate goes at least through one maximum is just a consequence of Rolle's theorem applied to the difference between the forward rate and the spot rate.

The property that the forward rate must be decreasing just before the spot rate has reached its maximum (or its plateau) can now be demonstrated as follows. Consider an increasing spot rate. It means that the last forward rate is *above* it: this is what has made the spot rate increase. Now if the spot rate, for the next maturity, does *not* change (because it has reached a maximum, or a plateau), the forward rate *must* be decreasing. Indeed, suppose the contrary: if the forward rate increases *or* stays constant, the spot rate will still increase, entailing a contradiction with our hypothesis. Therefore, the forward rate can only decrease. Theorem 2 (to follow in the text) has a similar interpretation.

instance T_0 . For this value of the abscissa, the differential $\frac{ds}{dT}(0, T_0)T_0$ is equal to distance δ drawn on the ordinate (Figure 3). Adding δ to the spot rate $s(0, T_0)$ yields the forward rate $f(0, T_0)$.

Conversely, from any given forward curve $f(0, T)$ the spot rate can be sketched in the following way. From point T_0 on the abscissa draw the vertical with ordinate $f(0, T_0)$. Consider a series of horizontal lines of length T_0 between height $f(0, T_0)$ and the abscissa. Each of those lines delineates two areas: one below the forward curve and above the horizontal; the other one below the horizontal and above the forward curve. The height of the horizontal for which both areas (A_2 and A_1 in our diagram) are equal yields the spot rate $s(0, T_0)$ (you have just applied the property $s(0, T_0)T_0 = \int_0^{T_0} f(0, z)dz$). Furthermore, not only do you get a fair idea of the spot rate, but of the slope of the spot curve as well at that point. Indeed, consider the difference δ between the forward curve and the spot curve at T_0 ; subtract δ from $s(0, T_0)$ on the ordinate. The line between that point and $(T_0, s(0, T_0))$ is the tangent of the spot curve at T_0 .

But it would be more rewarding to verify with precision the property we developed under Theorem 1. To that effect, we will proceed analytically and consider the spot rate curve as a spline⁶ represented by a third order polynomial starting at point $(0, 0.05)$ and ending at point $(20, 0.0929)$ ⁷, with a slope equal to zero at that end point.

The reason for which practitioners use third order polynomials for splines is little known, and merits mentioning here. It stems from the following important property. If a polynomial represents a lath joining a given set of points, a third-order polynomial minimizes both the curvature of the lath and its deformation energy. Indeed, either of these concepts is basically represented – within the confines of affine transformations – by the functional

$$I[s(z)] = \int_a^b [s''(z)]^2 dz \quad (32)$$

where $s''(z)$ is the second derivative of the spot rate – or of the spline (denoted $s(z)$ for convenience).

⁶On splines, see C. de Boor, *A Practical Guide to Splines* (New York: Springer Verlag, 1978) and G. Wahba, *Spline Models for Observational Data* (Philadelphia: SIAM, 1990).

⁷We have chosen those values arbitrarily. Of course, our qualitative results are independent of this choice.

Figure 3. The continuously compounded total return, the spot rate and the instantaneous forward rate.

Minimizing the integral I is a particular case of minimizing a functional

$$J[s(z)] = \int_a^b F[z, s(z), s'(z), s''(z)] dx.$$

where $s''(z)$ is the second derivative of the spot rate – or of the spline (denoted $s(z)$ for convenience).

Such a problem is solved by the Euler-Poisson 4th order differential equation

$$\frac{\partial F}{\partial s}(z, s, s', s'') - \frac{d}{dz} \frac{\partial F}{\partial s'}(z, s, s', s'') + \frac{d^2}{dz^2} \frac{\partial F}{\partial s''}(z, s, s', s'') = 0. \quad (33)$$

Notice that in our case (equation (32)) the arguments z, s and s' are missing in F , which depends solely upon s'' . So the Euler-Poisson equation (33) boils down to

$$\frac{d^2}{dz^2} \frac{\partial F}{\partial s''}(s'') = 0. \quad (34)$$

Applying (34) to (32), we have

$$\frac{\partial F}{\partial s''}(s'') = 2s''(z) \quad (35)$$

and therefore a first-order condition for $s(z)$ to minimize the functional I is that it solves the 4th order differential equation

$$\frac{d^2}{dz^2} [2s''(z)] = 2s^{(4)}(z) = 0, \quad (36)$$

which leads to the third-order polynomial

$$s(z) = \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0. \quad (37)$$

We will now adjust this polynomial to the curve depicted in Figure 3. To that effect, let us revert to our notation

$$s(0, T) = \alpha_3 T^3 + \alpha_2 T^2 + \alpha_1 T + \alpha_0. \quad (37a)$$

Since the spot rate curve is supposed to level off at its end point, only three points will be needed to adjust the curve: indeed, together with the first condition this will imply four equations in the four unknowns $\alpha_0, \alpha_1, \alpha_2$ and α_3 . The first three equations are determined by the knowledge of the initial

point, any intermediate point and the end point. Let these three points be denoted as

$$\begin{aligned}(T_0, s(0, T_0)) &\equiv (T_0, s_0) \\ (T_1, s(0, T_1)) &\equiv (T_1, s_1) \\ (\bar{T}, s(0, \bar{T})) &\equiv (\bar{T}, \bar{s}).\end{aligned}$$

Constraining the spot curve to go through these three points leads to the first three equations of system (38); the last equation of (38) reflects the fact that the derivative of the spot curve at $T = \bar{T}$ is equal to zero. The system of four equations in the four unknowns $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is:

$$\begin{aligned}\alpha_3 T_0^3 + \alpha_2 T_0^2 + \alpha_1 T_0 + \alpha_0 &= s_0 \\ \alpha_3 T_1^3 + \alpha_2 T_1^2 + \alpha_1 T_1 + \alpha_0 &= s_1 \\ \alpha_3 \bar{T}^3 + \alpha_2 \bar{T}^2 + \alpha_1 \bar{T} + \alpha_0 &= \bar{s} \\ 3\alpha_3 \bar{T}^2 + 2\alpha_2 \bar{T} + \alpha_1 &= 0\end{aligned}\tag{38}$$

Using the simplifying notation

$$\alpha = (\alpha_3, \alpha_2, \alpha_1, \alpha_0)'$$

$$s = (s_0, s_1, \bar{s}, 0)'$$

$$\mathbf{T} = \begin{bmatrix} T_0^3 & T_0^2 & T_0 & 1 \\ T_1^3 & T_1^2 & T_1 & 1 \\ \bar{T}^3 & \bar{T}^2 & \bar{T} & 1 \\ 3\bar{T}^2 & 2\bar{T} & 1 & 0 \end{bmatrix}$$

the vector of unknowns α of the polynomial is such that

$$\mathbf{T} \cdot \alpha = s\tag{39}$$

and therefore α is equal to

$$\alpha = \mathbf{T}^{-1} \cdot s\tag{40}$$

provided $\mathbf{T}$ is not singular.

In the example of Figure 3, the three points are⁸:

⁸These are rounded values. The actual ordinates s_1 and $\bar{s}_2$ used in the calculations were 0.071241 and 0.092857. Throughout this study all results are rounded, but the actual calculations are carried out with 20 decimals.

$$\begin{aligned}
(T_0, s_0) &= (0, 0.05) \\
(T_1, s_1) &= (5, 0.0712) \\
(\bar{T}, \bar{s}) &= (20, 0.0929)
\end{aligned}$$

which leads to the spot rate curve

$$\begin{aligned}
s(0, T) &= 2.21(10^{-6})T^3 - 0.000196T^2 + 0.005171T + 0.05 \quad \text{if } T \leq \bar{T} \\
s(0, T) &= 0.0929 \quad \text{if } T > 20
\end{aligned} \tag{41}$$

from which the following forward rate results:

$$\begin{aligned}
f(0, T) &= 8.86(10^{-6})T^3 - 0.000587T^2 + 0.01034T. \\
f(0, T) &= 0.0929 \quad (T > 20)
\end{aligned} \tag{42}$$

It is immediate to show that $f(0, T)$ has indeed the kind of behavior that was forecast. It goes through a *maximum* at an abscissa $\hat{T} = 12.146$; from $\hat{T}$ to $\bar{T}$ it is a *decreasing* function of maturity.

Figure 3 illustrates the behavior of the forward curve corresponding to the spot curve, together with the total return curve.

In accordance with (41) and (42), the total return is

$$\begin{aligned}
S(0, T) &= s(0, T) \cdot T = \int_0^T f(0, z) dz \\
&= \begin{cases} 2.21(10^{-6})T^4 - 0.000196T^3 + 0.005171T^2 + 0.05T & (\text{if } 0 \leq T \leq 20) \\ 0.0928T & (\text{if } T \geq 20) \end{cases}
\end{aligned} \tag{43}$$

Continuously increasing, with a continuous first derivative, the total return has an inflection point at $\hat{T} = 12.146$.

A word of caution is in order here: we have shown that the forward rate curve went at least through one maximum for $T < \bar{T}$, but we have said nothing about its concavity or convexity. In the example of Figure 3, the forward curve implied by the concave spot curve turned out to be concave throughout, but this is not at all a general rule. Local convexity can be the case if the following occurs. Consider the derivative of (30). We get

$$\frac{d^2 f(0, T)}{dT^2} = 3 \frac{d^2 s(0, T)}{dT^2} + \frac{d^3 s(0, T)}{dT^3} T. \tag{44}$$

At the origin ($T = 0$) a concave spot curve ($d^2s(0,0)/dT^2 < 0$) will always entail a concave forward curve ($d^2f(0,0)/dT^2 < 0$). But for $T > 0$, the forward curve will be convex ($d^2f(0,T)/dT^2 > 0$) if and only if

$$\frac{d^3s(0,T)}{dT^3} > -\frac{3}{T} \frac{d^2s(0,T)}{dT^2}. \quad (45)$$

which implies, since $d^2s(0,T)/dT^2 < 0$, $\frac{d^3s(0,T)}{dT^3} / \frac{d^2s(0,T)}{dT^2} < -\frac{3}{T}$.

Denote

$$\frac{d^3s(0,T)}{dT^3} / \frac{d^2s(0,T)}{dT^2} = s'''/s'' = \frac{d \ln s''}{dT} \equiv c$$

the *relative rate of increase* in concavity of $s(0,T)$. Condition (45) just amounts to $c < -3/T$.

It is quite interesting to observe how sensitive this condition is to a very small variation in the spot rate. We will suppose that one point only of our original spline is modified by changing point (5, 0.0712) into (5, 0.0747). The spline, always concave, becomes

$$s(0,T) = 5.3(10^{-6})T^3 - 0.00032T^2 + 0.0064T + 0.05 \quad (46)$$

and the forward curve becomes

$$f(0,T) = 0.000021T^3 - 0.00096T^2 + 0.01282T + 0.05 \quad (47)$$

This time the forward rate goes through an inflection point at $T^* = 15.045$ years⁹, and becomes convex between T^* and T , as Figure 4 shows. Only precise calculations of condition (45) lead to this conclusion, which cannot be obtained by simple inspection of the spot rate curve.

We will now close this section by drawing attention upon a rather surprising consequence of Theorem 1. The usual, innocuous looking, increasing and concave spot rate curve covers a well-hidden, important message regarding the efficiency of investments as a function of their *horizon*. Indeed, it implies a total return on investments which has the exact outlook of a one-variable S -shaped production function with an inflection point, where the variable is neither capital nor labor, but the investment's horizon (see Theorem 1 in §1.5, illustrated in Figure 3). For short and medium term projects, this

⁹This corresponds to the solution of equation $c = s'''(0,T)/s''(0,T) = -3/T$.

Figure 4. A concave spot rate curve does not necessarily entail a concave forward rate curve

marginal efficiency is first increasing; it is then decreasing. For very long horizons the horizontal spot rate curve, equal to the forward rate, makes the case of constant returns.

These observations prompt us to indulge in the following conjecture. An increasing marginal efficiency of short and medium term investments followed by decreasing marginal efficiency of longer investments may help track down the ever-elusive causes of the “normal” shape of the interest rate term structure. It is our belief that theories of the interest term structure may have been somewhat lopsided until now, centering essentially on the behavior of agents in the financial markets, thus giving short shrift to its real counterpart. A natural question to ask now is whether the increasing and decreasing marginal efficiency of investments according to their horizon stands the trial of facts.

2 Immunization: a First Approach

In the introduction of this chapter, we have shown the essential role of the investor’s horizon when evaluating the consequences of any change in the rates of interest. We had in mind two objects of interest: first, the instantaneous change in the bond portfolio’s value due to the drop or the rise of interest rates; and second, the relative decrease or increase, respectively, of the reinvestment proceeds of the coupons. We can now be more specific, and define the future value, at horizon H , of the portfolio: this will take into account both effects at the same time.

However, this measure of the future bond performance can still be improved upon the following reason. A future value, in itself, is not easy to picture. For instance, an investment’s future value equal to 136, in $H = 6$ years, when the present value is 94, is difficult to interpret. It is much more convenient to read the investment’s performance in terms of a yearly rate of return, either compounded once a year, or continuously compounded. If compounded once a year, the yearly return is $(136/94)^{1/6} - 1 = 6.35\%$; continuously compounded, the yearly return is $(1/6) \ln(136/94) = 6.16\%$. We will now introduce in a precise way the horizon rate of return. We have at least two choices: we can consider it either compounded once a year, or continuously compounded. As the reader will appreciate later, continuously compounded returns are much more convenient to deal with than any other form of return, so we will use those.

In order to have a clear idea of immunization, we will first suppose that the spot rate structure is flat, and that it undergoes a parallel shift. We can use the tools we are now familiar with, and set in the spot rate polynomial

$$s(0, T) = a_1 + a_2T + a_3T^2 + a_4T^3 \quad (48)$$

$a_2 = a_3 = a_4 = 0$, retaining only a_1 . So $s(0, T) = a_1$ at time 0 when a bond or a bond portfolio is bought. A very short time later, at time ε , the spot structure shifts to a new value A_1 , which may be higher or lower than a_1 . The portfolio's original value, denoted $B(a_1)$, now changes to $B(A_1)$. In terms of the horizon H , its new (future) value, denoted $B_H(A_1)$, is:

$$B_H(A_1) = B(A_1) \exp(A_1 H) \quad (49)$$

2.1 The continuously compounded horizon rate of return

The continuously compounded H -horizon rate of return, r_H , transforms an initial investment value, $B(a_1)$, which undergoes an instantaneous change in the rates of interest, into a future value at horizon H . Let us denote it r_H . Using (49), r_H is such that

$$B(a_1)e^{r_H H} = B_H(A_1) = B(A_1)e^{A_1 H} \quad (50)$$

and hence

$$r_H = \frac{1}{H} \log[B(A_1)/B(a_1)] + A_1 \quad (51)$$

The horizon rate of return, given a_1 , is a function of the two variables A_1 and H . It can thus be viewed as a family of curves in (r_H, A_1) space where H plays the role of a parameter. Those parametric curves are written $r_H = r_H(H; A_1)$.

Before rushing to a diagram representing this family of curves, let us think about the first messages conveyed by formula (51). One of the most immediate is quite remarkable: if interest rates do not change, the horizon rate of return is *independent* from the horizon. Indeed, set $A_1 = a_1$ in (51); r_H then equals a_1 and is independent of H . So we already know that all curves $r_H(H; A_1)$ will go through the fixed point (a_1, a_1) . This is not at all obvious: think of 8.75 percent coupon bond, with maturity 22.3 years;

consider a horizon of 6 years. It is not evident that if the spot rate remains at 7.25 percent the 6-year horizon rate of return will be 7.25. Only a lengthy reasoning through arbitrage on the bond's value would provide that simple property.

The second message provided by (51) is that when H tends to infinity, the horizon rate of return becomes identical to A_1 , i.e. $\lim_{H \rightarrow \infty} r_H = A_1$. In (r_H, A_1) space, r_H becomes the linear function A_1 . That has an immediate interpretation: whatever initial capital gain or loss the bond has incurred because a_1 moved to A_1 , in the long run only rate A_1 prevails as a determinant of the investor's return.

2.2 A geometrical representation of the horizon rate of return

Now we may want to think of the nature of the curves we must get. We already know that all curves will go through point (a_1, a_1) . Additionally, we may surmise that short horizon curves must be decreasing. The reason is that in the short run capital gains due to a decrease in a_1 far outweigh the losses entailed by coupon reinvestment at a lower rate, and the converse is true in the case of increasing rates. Also, long horizon curves must be increasing, for symmetrical reasons.

We may well venture to guess how any of those curves moves when parameter H increases. Definitely, it must move counter-clockwise. The reason is subtle, and is two-fold. Think how a 2-year horizon curve must behave with regard to a 1-year horizon curve. For values of A_1 smaller than a_1 , the 2-year curve must be lower than the 1-year curve: first, the initial capital gain entailed by the spot rate decrease is more eroded after 2 years than after one year; and second, the coupon is reinvested at a lower rate during a longer time. The contrary applies when the rates increase; so definitely the curves will pivot counter-clockwise around the fixed point (a_1, a_1) . This is indeed confirmed in figure 5, where the horizon rate of return is depicted for the holder of a 20-year, 8 percent bond, for horizons equal to 1, 2, 5, 15 years, and for an infinite horizon. The curves look like straight lines, although they are not (with the exception of $r_{H=\infty} = A_1$). Each displays a slight convexity, as we will show later.

The natural question to ask now is whether there exists an intermediate horizon such that the rate of return for that horizon is an (almost flat) curve

Figure 5. The family of horizon rates of return $r_H(A_1)$ for various horizons.

going through (a_1, a_1) . From an economic point of view, it implies that any capital gain (or loss) due to a change in interest rates is compensated by a corresponding loss (or gain) in the reinvestment of coupons. Should this prove to be the case, it would imply that the investor with such a horizon would be protected against any unforeseen change in the interest rates. It turns out that this horizon exists, and that it corresponds to an essential feature of the bond; we now tackle this question of central importance.

2.3 Existence and characteristics of an immunizing horizon

For such an immunizing horizon to exist, H must be such that r_H goes through a minimum at point (a_1, a_1) . A sufficient condition is that $r'_H(a_1) = 0$ and $r''_H(a_1) > 0$. The first-order condition implies

$$r'_H(a_1) = \frac{1}{H} \frac{d \log B}{dA_1} \Big|_{A_1=a_1} + 1 = 0 \quad (52)$$

and hence

$$H = - \frac{d \log B}{dA_1} \Big|_{A_1=a_1}. \quad (53)$$

Thus the horizon should be equal to minus the logarithmic derivative of the bond's value with respect to A_1 at point a_1 . Denoting c_t ($t = 1, \dots, N$) the cashflows of the bond, we have

$$B(A_1) = \sum_{t=1}^N c_t e^{-A_1 t} \quad (54)$$

and

$$H = - \frac{d \log B(A_1)}{dA_1} \Big|_{A_1=a_1} = - \frac{1}{B(A_1)} \frac{dB(A_1)}{dA_1} \Big|_{A_1=a_1} = \sum_{t=1}^N t c_t e^{-a_1 t} / B(a_1). \quad (55)$$

The last expression in (55) is the weighted average of the times of payment of the bond, the weights being the shares of the present value cashflows in the bond's value. This average is called the *duration* of the bond and is denoted

D. It is of central importance in bond immunization¹⁰. Invented by Frederik Macaulay (1938), it bears the name “Macaulay duration” if a is either the spot interest rate, constant for all maturities, or equivalently the bond’s yield to maturity. If the cash flows are discounted with a non-constant spot rate structure, it bears the name “Fisher-Weil duration”, from Lawrence Fisher and Roman Weil’s 1971 paper.

It should be noted that the duration of a bond portfolio is the weighted average of the durations of the bonds comprising the portfolio, the weights being the shares of each bond of the portfolio. The demonstration will be made in §3.6.3, in a more general context. Thus a first-order condition for an investment in bonds to be immunized against changes in the interest rates is that the horizon is equal to the duration of the bond.

Let us now check that the second-order condition, $r_H''(a_1) > 0$, is met. Taking the derivative of (52) with respect to A_1 gives:

$$r_H'' = \frac{d}{dA_1} \left[\frac{1}{H} \frac{B'}{B} \right] = \frac{1}{H} \left[\frac{B''B - B'^2}{B^2} \right] = \frac{1}{H} \left[\frac{B''}{B} - \left(\frac{B'}{B} \right)^2 \right] \quad (56)$$

Evaluating $B''(A_1)/B(A_1)$, and denoting $B(A_1)$ as B , we have:

$$B''/B = \frac{1}{B} \frac{d}{dA_1} \left[- \sum_{t=1}^N t c_t e^{-A_1 t} \right] = \sum_{t=1}^N t^2 c_t e^{-A_1 t} / B(A_1). \quad (57)$$

Plugging this result into (56) and using the simplifying notation $c_t e^{-A_1 t} / B = w_t$, we can write r_H'' as:

$$r_H'' = \frac{1}{H} \left[\sum_{t=1}^N t^2 w_t - \left(\sum_{t=1}^N t w_t \right)^2 \right]. \quad (58)$$

The bracketed term is just the variance of the times of payment, which is always positive. Thus the second order condition for a global minimum of r_H is met¹¹. This is confirmed in our example. Our bond has a duration

¹⁰For a general presentation of this concept and its properties, we take the liberty to refer the reader to our book.

¹¹It is useful to check the units of our formula (58). First, r_H is in (1/year), so $r_H' = dr_H/dA_1$ is unitless since A_1 is in (1/year). Therefore $r_H'' = dr_H'/dA_1$ must be in 1/(1/year) = years. This is the case: the right-hand side of (58) is in (years²/year) = years.

$D = \sum_{t=1}^N t c_t e^{-a_1 t} / B(a_1) = 11.89$ years. The curve $r_{H=D} = 11.89$ goes through a minimum at point a_1 . Note also that the global convexity of r_H enables to confirm that all curves in figure 5 are strictly convex (except $r_{H=\infty} = A_1$).

As the reader can appreciate, these result immediately extend from a given bond to a bond portfolio, because the formula $\sum_{t=1}^N c_t w_t$ for the value of a bond immediately encompasses that of a bond portfolio: c_t stands for any fixed income received at time t , and it may well correspond to a bond portfolio.

Until now we have considered the problem of finding a horizon H such that a bond, or a bond portfolio, would be immune to variations of the horizontal spot structure. We can now ask the converse question: given a horizon H , is it possible to *build* a portfolio such that its horizon- H rate of return (equivalently: its future value) would be immune to a variation of that structure. And we can immediately generalize the question to the construction of a bond portfolio whose future value would be protected against *any* variation of *any* initial spot rate structure.

2.4 A first generalization

We can see easily how this simple case is nothing but a particular case of the more general setting in which the spot rate for maturity t , $s(t)$, is expressed as an $m-1$ order polynomial $s(t) = a_1 + a_2 t + a_3 t^2 + \dots + a_m t^{m-1}$. The total return would be

$$S(t) = s(t)t = a_1 t + a_2 t^2 + \dots + a_m t^m$$

and the values of the portfolio would be a function of the vector ($\mathbf{a} = a_1, \dots, a_m$) instead of the scalar a_1 . Suppose however that, among the elements of this vector, a_1 only is allowed to move: this implies that the spot rate curve $s(t)$ is shifted by a constant, i.e. it moves parallel to itself.

Since only a_1 is allowed to move to a new value A_1 , the horizon rate is still given by (51); the only difference is that in the right-hand side we have

$$B(A_1) = \sum_{t=1}^m c_t e^{-(A_1 t + a_2 t^2 + \dots + a_m t^m)} \quad (59)$$

and

$$B(a_1) = \sum_{t=1}^m c_t e^{-(a_1 t + a_2 t^2 + \dots + a_m t^m)} \quad (60)$$

Equation (52) again applies; as to minus the logarithmic derivative of $B(A_1)$ in (53), it is given by

$$-\frac{d \log B}{dA_1} \Big|_{A_1=a_1} = \sum_{t=1}^N t c_t e^{-(a_1 t + a_2 t^2 + \dots + a_m t^m)} / B(a) \quad (61)$$

where the expression on the right-hand side of (61) is defined as the duration of the bond portfolio.

There is no reason however why so simple scenarios would apply in reality: first, the spot rate structure is practically never represented by a horizontal; second, if considered as a polynomial, there is no reason why it should shift by a constant. Quite on the contrary, short rates are more volatile than longer ones. In consequence a broader analysis is called for, allowing for arbitrary changes in the spot rate structure. This is the question we address in Section 3.

3 Protecting investors against any shift in the interest rate structure – a general immunization theorem

We will determine the present and future values of a bond portfolio before and after the shift in the interest rate structure. We will then present some new tools which are of central importance in the immunization process: the moment of order k of a bond and of a bond portfolio, as well as fundamental properties of those moments. We will then be ready to state and prove our immunization theorem.

3.1 Notation

We will use the following notation:

$L \equiv$ number of different securities (bonds) in the bond portfolio bought at time 0; the bonds are labelled $l = 1, \dots, L$.

$n_l \equiv$ number of type l bonds in portfolio. The total number of securities in the portfolio is $\sum_{l=1}^L n_l$.

$c_{lt} \equiv$ nominal cash flow received by holder of bond l at time t ; in monetary units (for instance \$).

$c_t = \sum_{l=1}^L n_l c_{lt} \equiv$ total nominal cash flows received by holder of portfolio at time t ; in \$.

$s(t) \equiv$ the continuously compounded interest rate (per year) applying today, at time $t = 0$, for a loan to be reimbursed in t years (t not necessarily an integer). Expressed in (1/year). It is also called the continuously compounded *spot* rate at time 0 for maturity t .

$f(u) \equiv$ the forward rate agreed upon at time 0 for lending at time u ($u > 0$) for an infinitesimally short interval - equivalently: for "instantaneous" lending. Expressed in (1/year).

$S(t) \equiv$ the continuously compounded *total* return over a period t . It is equal to the integral sum of the forward rates over $[0, t]$, $\int_0^t f(u) du = s(t)t$. It is a pure (dimensionless) number.

$\mathbf{a} = (a_1, a_2, \dots, a_m) \equiv$ (line) vector of order m of Taylor's series expansion coefficients for $s(t)$ at time 0 when the portfolio is bought; $s(t) \approx a_1 + a_2 t + \dots + a_m t^{m-1}$. Note the measurement units of a_j : they are $1/(\text{year})^j$, $j = 1, \dots, m$. This observation will prove to be quite useful later in this Section. For the time being, observe that $s(t) \approx a_1 + a_2 t + \dots + a_m t^{m-1}$ is therefore expressed in (1/year) as it should.

$\mathbf{t}^p = (t^1, t^2, \dots, t^m) =$ (column) vector of m successive positive integer powers of t (the small "p" as a subscript of t stands as "powers").

$S(t) = s(t)t = \int_0^t f(u) du$ can be approximated by the following polynomial:

$$S(t) \approx a_1 t + a_2 t^2 + \dots + a_m t^m = \mathbf{a} \cdot \mathbf{t}^p$$

and is expressed as a pure number.

$B_0^l(\mathbf{a}) \equiv$ value of bond l at time 0 (in \$).

$P_0(\mathbf{a}) \equiv$ value of portfolio at time 0 (in \$).

$H \equiv$ immunization horizon, in years; not necessarily an integer number.

$K \equiv$ a positive constant; its dimension is $(\text{years})^{2m}$.

$\mathbf{h}^p = (H^1, H^2, \dots, H^m) \equiv m$ -component vector of successive positive integer powers of horizon H .

$\mathbf{h}_{2m+1}^p \equiv (2m+1)$ -component vector, equal to: $(1, H, H^2, \dots, H^{2m} + K)$.

$\varepsilon \equiv$ time, infinitesimally close to 0, at which the spot rate structure undergoes a variation.

$\mathbf{A} = (A_1, A_2, \dots, A_m) \equiv$ new vector of coefficients of $s(t)$'s Taylor expansion at time ε .

$B^{(l)}(\mathbf{A}) \equiv$ value of bond l at time ε , when the spot rate function $s(t)$ has undergone a variation.

$P_\varepsilon(\mathbf{A}) \equiv$ value of portfolio at time ε , when the $s(t)$ function has undergone a variation.

$B_H^{(l)}(\mathbf{A}) \equiv$ value of bond l at time ε in terms of year H dollars, when the $s(t)$ function has undergone a variation (future value of bond l at time ε).

$P_H(\mathbf{A}) \equiv$ value of portfolio at time ε in terms of year H dollars, when the $s(t)$ function has undergone a variation (future value of portfolio at time ε).

$N \equiv$ portfolio's maturity (in years); maximum maturity of bonds in the portfolio.

$r \equiv$ yield to maturity of bond portfolio; r is defined by the equality $P_0 = \sum_{t=1}^N c_t \exp(-rt)$.

$\sum_{t=1}^N tc_t \exp(-rt)/P_0 \equiv$ Macaulay duration of portfolio, evaluated with continuous compounding.

$\sum_{t=1}^N tc_t \exp[-s(t)t] \equiv$ Fisher-Weil duration $= \sum_{t=1}^N tc_t \exp(-\int_0^t f(u)du)$
 $= \sum_{t=1}^N tc_t \exp[-S(t)] = \sum_{t=1}^N tc_t \exp(-a.t^p) \equiv$ Fisher-Weil duration (in years).

We consider, for bonds and bond portfolios, basically four types of value:

- present value at time $t = 0$, the time of purchase (or in \$ of year 0).
- present value at time $t = \varepsilon$, an infinitesimally small instant after $t = 0$ (in \$ of year 0), when the spot rate structure has undergone a variation.
- future value at time $t = 0$ in terms of H -horizon dollars. One should be very clear about the expression “future value”, which may be a misnomer. It does *not* designate the value that the investment will have at time H . It is the amount one would get at $t = H$ by investing at time $t = 0$ the bond price at the spot rate $s(H)$ observed at $t = 0$.
- future value at time ε in terms of H -horizon dollars: the amount one would get at time H by investing at time ε (or, equivalently, at time 0 with the new structure) the bond price at the *new* spot rate. We say that both values are equivalent because ε is infinitesimally close to zero.

3.2 Present values at time 0

The present values, at time 0, are:

- for bond l ($l = 1, \dots, L$):

$$\begin{aligned} B_0^l(\mathbf{a}) &= \sum_{t=1}^N c_{lt} e^{-s(t)t} = \sum_{t=1}^N c_{lt} e^{-\int_0^t f(u)du} = \sum_{t=1}^N c_{lt} e^{-S(t)} \\ &= \sum_{t=1}^N c_{lt} e^{-(a_1 t + a_2 t^2 + \dots + a_j t^j + \dots + a_m t^m)} = \sum_{t=1}^N c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) \quad (62) \end{aligned}$$

- for the portfolio:

$$\begin{aligned} P_0(\mathbf{a}) &= \sum_{t=1}^N c_t e^{-s(t)t} = \sum_{t=1}^N c_t e^{-\int_0^t f(u)du} = \sum_{t=1}^N c_t e^{-S(t)} \\ &= \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) = \sum_{t=1}^N \sum_{l=1}^L n_l c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) \quad (63) \end{aligned}$$

3.3 Future values at time 0

The bonds and portfolios “future values”, expressed at time 0 in terms of H -year dollars, are, respectively:

$$\begin{aligned}
B_{0,H}^l(\mathbf{a}) &= B_0^l(\mathbf{a})e^{s(H)H} = B_0^l(\mathbf{a})e^{\int_0^H f(u)du} = B_0^l(\mathbf{a})e^{S(H)} \\
&= \sum_{t=1}^N c_{lt} e^{-(a_1 t + \dots + a_j t^j + \dots + a_m t^m)} e^{a_1 H + \dots + a_j H^j + \dots + a_m H^m} \\
&= \sum_{t=1}^N c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) \cdot \exp(\mathbf{a} \cdot \mathbf{h}^p) \\
&= \sum_{t=1}^N c_{lt} \exp[(-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p))] \tag{64}
\end{aligned}$$

Similarly, the portfolio's value in terms of horizon- H years is

$$P_{0,H} = \sum_{t=1}^N c_t \exp[(-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p))] = \sum_{t=1}^N \sum_{l=1}^L n_l c_{lt} \exp[(-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p))] \tag{65}$$

3.4 Present values at time ε

Suppose now that at time ε the spot rate function $s(t)$ has undergone a variation (which may be denoted $\eta(t)$), such that its new approximation as a Taylor expansion is:

$$s_\varepsilon(t) = A_1 + A_2 t + \dots + A_j t^{j-1} + \dots + A_m t^{m-1}. \tag{66}$$

We then have:

$$S_\varepsilon(t) = A_1 t + A_2 t^2 + \dots + A_j t^j + \dots + A_m t^m = \mathbf{A} \cdot \mathbf{t}^p. \tag{67}$$

The new present and future values formulas for the bond and the bond portfolio keep the same form as above, except that vector $\mathbf{A}$ now replaces vector $\mathbf{a}$; the present values are:

$$B_\varepsilon^l(A) = \sum_{t=1}^N c_{lt} \exp(-\mathbf{A} \cdot \mathbf{t}^p) \tag{68}$$

and

$$P_\varepsilon(A) = \sum_{t=1}^N c_t \exp(-\mathbf{A} \cdot \mathbf{t}^p) = \sum_{t=1}^N \sum_{l=1}^S n_l c_{lt} \exp(-\mathbf{A} \cdot \mathbf{t}^p). \tag{69}$$

3.5 Future values at time ε

Under the new spot rate structure, the future values are, at time ε :

$$B_{\varepsilon,H}^l(A) = \sum_{t=1}^N c_{lt} \exp[-A \cdot (t^p - h^p)] \quad (70)$$

and

$$P_{\varepsilon,H}(A) = \sum_{t=1}^N c_t \exp[-A \cdot (t^p - h^p)] = \sum_{t=1}^N \sum_{l=1}^S n_l c_{lt} \exp[-A \cdot (t^p - h^p)]. \quad (71)$$

3.6 Further Concepts for Immunization: The Moments of Order k of a Bond and a Bond Portfolio

We will now introduce some little known, but essential, concepts: the moment of order k of a bond or a bond portfolio. This concept is very much akin to the moment of a random variable, with two main differences: the objects for which the calculation of moments are made are not random, but deterministic, and their values are always positive.

3.6.1 The moment of order k of a bond

• Definition: The *moment of order k of a bond* is the weighted average of the k th power of its times of payments, the weights being the shares of the bond's cash flows in present value in the bond's present value. For bond l , this moment of order k is denoted $\mu_k^{(l)}$, and is equal to:

$$\mu_k^{(l)} = \sum_{t=1}^N t^k c_{lt} e^{-s(t)t} / B_0^l = \sum_{t=1}^N t^k c_{lt} e^{-\int_0^t f(u)du} / B_0^l = \sum_{t=1}^N t^k c_{lt} e^{-S(t)} / B_0^l, \quad k = 0, 1, 2, \dots$$

and, using our assumption that about $s(t)$:

$$\mu_k^{(l)} = \sum_{t=1}^N t^k c_{lt} \exp(-a \cdot t^p) / B_0^l(a), \quad k = 0, 1, 2, \dots \quad (72)$$

A moment of order k is expressed in (year) ^{k} units.

3.6.2 The moment of order k of a bond portfolio

• Definition: The moment of order k of a bond portfolio is the weighted average of the k th power of its times of payments, the weights being the shares of the portfolio's cashflows in present value in the portfolio's present value. Denoted $\mu_k^{(P)}$, it is equal to:

$$\mu_k^{(P)} = \sum_{t=1}^N t^k c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}), \quad k = 0, 1, 2, \dots \quad (73)$$

Note that the moment of order 0 of a portfolio (or a bond) is one, since it is the weighted average of 1's. Also, the moment of order one of a portfolio (or a bond) is its Fisher-Weil duration.

3.6.3 Properties of moments

We will first show the relationship between the moments of order k of a portfolio and the moments of same order of the bonds making up the portfolio. We will then show how these moments measure sensitivities of the bond prices to changes in the coefficients of the $s(t)$'s polynomial.

3.6.3.1. Theorem 3

The k th moment of a bond portfolio is the weighted average of the k th moments of the bonds making up the portfolio; the weights are the shares of each bond in the portfolio.

3.6.3.2. Proof

From (73) we can write successively:

$$\begin{aligned} \mu_k^{(P)} &= \sum_{t=1}^N t^k c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}) = \sum_{t=1}^N t^k \sum_{l=1}^L n_l c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}) \\ &= \sum_{l=1}^L \sum_{t=1}^N t^k n_l c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}) = \sum_{l=1}^L n_l \sum_{t=1}^N t^k c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}) \\ &= \sum_{l=1}^L \frac{n_l B_0^l(\mathbf{a})}{P_0(\mathbf{a})} \sum_{t=1}^N t^k c_{lt} \exp(-\mathbf{a} \cdot \mathbf{t}^p) / B_0^l(\mathbf{a}). \end{aligned} \quad (74)$$

If $\alpha_l = \frac{n_l B_0^l(\mathbf{a})}{P_0(\mathbf{a})}$ designates the share of bond l 's value in the initial portfolio ($\sum_{l=1}^L \alpha_l = 1$), then:

$$\mu_k^{(P)} = \sum_{l=1}^L \alpha_l \mu_k^{(l)}, \quad k = 0, 1, 2, \dots \quad (75)$$

3.6.3.3. Theorem 4 *Minus the logarithmic derivative of the l bond's [the portfolio's] price with respect to coefficient a_j is equal to the j th moment of the bond [the portfolio].*

3.6.3.4. Proof We will prove this theorem for a bond portfolio, a single bond being just a particular case.

From (63):

$$P_0(\mathbf{a}) = \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^P).$$

Its logarithmic derivative (its relative rate of increase in linear approximation) with respect to a_j is

$$\frac{1}{P_0(\mathbf{a})} \frac{\partial P_0(\mathbf{a})}{\partial a_j} = - \sum_{t=1}^N t^j c_t \exp(-\mathbf{a} \cdot \mathbf{t}^P) / P_0(\mathbf{a}) \quad (j = 1, \dots, m). \quad (76)$$

Applying (73), we recognize in the right-hand side of (76) minus the j th moment of the portfolio. Thus we have the fundamental relationship contained in Theorem 4:

$$\frac{-1}{P_0(\mathbf{a})} \frac{\partial P_0(\mathbf{a})}{\partial a_j} = \mu_j^{(P)} \quad (j = 1, \dots, m). \quad (77)$$

Three remarks are in order at this point. First, note that the dimensions of both sides of (77) do match, as they should: from our observation in §3.1 on the measurement units of a_j ($j = 1, \dots, m$), the left-hand side of (77) is expressed in $1/[1/(\text{year})^j] = \text{year}^j$. These are indeed the units of the j th moment of the portfolio, as noted above.

Second, suppose that the polynomial $s(t)$ reduces to a_1 . We then have $s(t) = a_1 \equiv r$ (a constant), and $S(t) = s(t)t = \int_0^t f(u)du = \int_0^t a_1 du = a_1 t$.

The portfolio's value reduces to $P_0(a_1) = \sum_{t=1}^N c_t \exp(-a_1 t)$; a_1 is nothing else than the portfolio's yield to maturity, equal to r , by the definition of the yield to maturity given in §3.1.

We then have:

$$\frac{-1}{P_0(a_1)} \frac{dP_0(a_1)}{da_1} = \sum_{t=1}^N t c_t \exp(-a_1 t) / P_0(a_1). \quad (78)$$

This reduces to the well-known property that the logarithmic derivative of the portfolio's value with respect to the constant $a_1 = r$ (or the yield to maturity) is equal to minus its Macaulay duration evaluated with continuous compounding.

Another particular case is of interest. Suppose that the spot rate structure does not reduce to a constant, but that it retains its general form of an $(m-1)$ th order polynomial. However, consider that only coefficient a_1 moves: this implies that the whole structure (which is not a constant) is shifted up or down by a given amount. In other words, the structure moves in a parallel fashion. Taking the logarithmic partial derivative of (63) with respect to a_1 (or, equivalently, replacing a_j by a_1 in (76)), we get:

$$\frac{-1}{P_0(\mathbf{a})} \frac{\partial P_0(\mathbf{a})}{\partial a_1} = \sum_{t=1}^N t c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P_0(\mathbf{a}). \quad (79)$$

This time our particular case amounts to the equality between the portfolio's sensitivity to a *parallel* change in the spot rate structure and the Fisher-Weil duration.

3.7 A general immunization theorem

We will now state and prove the following immunization theorem.

3.7.1 Theorem 5

Suppose that the spot rate structure can be expanded into a Taylor series of order $m-1$, and that it undergoes a variation. Then a sufficient condition for a bond portfolio to be immunized against such a variation is the following:

- *any moment of order k ($k = 0, 1, \dots, 2m-1$) of the bond portfolio is equal to the k th power of the investor's horizon H and*
- *the moment of order $2m$ is equal to the $2m$ th power of H plus a positive, arbitrary constant.*

any moment of order k ($k = 0, 1, \dots, 2m - 1$) of the bond portfolio is equal to the k th power of the investor's horizon H and

the moment of order $2m$ is equal to the $2m$ th power of H plus a positive, arbitrary constant.

3.7.2 Proof

Suppose that the initial spot rate structure $s(t)$ can be approximated to the $(m - 1)$ th order by the following polynomial

$$s(t) \approx a_1 + a_2 t + a_3 t^2 + \dots + a_j t^{j-1} + \dots + a_m t^{m-1} \quad (80)$$

where $a_j = \frac{1}{(j-1)!} s^{(j-1)}(0)$, $j = 1, \dots, m$; $s^{(j-1)}(0)$ is the $(j - 1)$ th derivative of $s(t)$ at $t = 0$.

The total return $S(t) = s(t)t$ is therefore approximated by

$$S(t) = a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_j t^j + \dots + a_m t^m = \mathbf{a} \cdot \mathbf{t}^p. \quad (81)$$

The implied forward rate function is then:

$$f(t) = S'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \dots + j a_j t^{j-1} + \dots + m a_m t^{m-1}. \quad (82)$$

(The reader may verify on this example one of the properties we had demonstrated under §1.5, namely that at the origin the slope of the forward rate curve is twice that of the spot rate curve. Indeed, at the origin the slope of the spot rate curve is $s'(0) = a_2$, while $f'(0) = 2a_2$).

The value of a bond portfolio bought at time 0, under the initial spot rate structure, is thus

$$P_0(a) \approx \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p).$$

Under the initial spot structure $\mathbf{a} \cdot \mathbf{t}^p$, the portfolio's *future* value, at horizon H is:

$$P_{0,H}(a) \approx P_0(a) e^{a_1 H + a_2 H^2 + \dots + a_m H^m} = P_0(a) \exp(\mathbf{a} \cdot \mathbf{h}^p)$$

where $\mathbf{h}^p$ denotes the vector $(H, H^2, \dots, H^m)$. Therefore

$$P_{0,H}(a) = \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) \exp(\mathbf{a} \cdot \mathbf{h}^p) = \exp(\mathbf{a} \cdot \mathbf{h}^p) \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p).$$

Suppose now that the whole spot rate structure $s(t)$ undergoes a variation: from $\mathbf{a} \cdot \mathbf{t}^p$ at time 0 it becomes $\mathbf{A} \cdot \mathbf{t}^p$ at time ε . Therefore the future value of the bond portfolio becomes

$$P_{\varepsilon,H} = \left[\sum_{t=1}^N c_t \exp(-\mathbf{A} \cdot \mathbf{t}^p) \right] \exp(\mathbf{A} \cdot \mathbf{h}^p). \quad (83)$$

For the portfolio to be immunized, a first order condition is that the gradient of any positive transformation of $P_{\varepsilon,H}$ equals zero. Taking logs on both sides of (83):

$$\log P_{\varepsilon,H} = \mathbf{A} \cdot \mathbf{h}^p + \log \left[\sum_{t=1}^N c_t \exp(-\mathbf{A} \cdot \mathbf{t}^p) \right]. \quad (84)$$

Differentiate (84) with respect to the m variables $A_1, \dots, A_j, \dots, A_m$, and equate to zero these derivatives at point $\mathbf{a}$:

$$\frac{\partial \log P_{\varepsilon,H}(\mathbf{a})}{\partial A_j} = H^j + \frac{-1}{P_0} \left[\sum_{t=1}^N t^j c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) \right] = 0, \quad j = 1, \dots, m. \quad (85)$$

These m equations (85) imply that each j th moment of the portfolio must be equal to the j th power of the investor's horizon:

$$\mu_j^{(P)} = H^j, \quad j = 1, \dots, m \quad (86)$$

A first set of conditions is defined by the $m + 1$ equations corresponding to:

a) the m equations implied by (86)

b) an additional equation corresponding to the accounting constraint: the sum of all shares of each bond in the portfolio must be equal to one.

We will take up these conditions in order, together with their implications in terms of the amounts of the bonds to be chosen.

For (a) to be met, let us use the property that the portfolio's moment of order j is the weighted average of each bond's moment of order j , the weights being the share of each bond in the portfolio:

$$\mu_j^{(P)} = \sum_{l=1}^L \frac{n_l B_0^l(\mathbf{a})}{P_0(\mathbf{a})} \mu_j^{(l)}, \quad j = 1, \dots, m. \quad (87)$$

A word of caution is in order here. It would be tempting to solve the above system of $m + 1$ equations in the $m + 1$ unknowns $n_1, \dots, n_{m+1}$ and consider the solution as a possible solution for the immunizing portfolio. This system reflects the first order conditions only and not the second-order ones. We will show that we can never form an optimal portfolio by relying on first-order conditions only.

Let us now address second-order conditions and their implications. For a local minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$, to equations (90) we have to add those corresponding to a positive semi-definite Hessian matrix for $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. We will now tackle these conditions.

Recall that the portfolio's future value at horizon H and at time ε is given by (83), which we can write equivalently as:

$$P_{\varepsilon;H}(\mathbf{A}) = \sum_{t=1}^N c_t \exp[-\mathbf{A} \cdot (\mathbf{t}^p - \mathbf{h}^p)]. \quad (91)$$

Taking the partial derivative of (91) with respect to A_j ($j = 1, \dots, m$) yields

$$\frac{\partial P_{\varepsilon;H}}{\partial A_j}(\mathbf{A}) = - \sum_{t=1}^N c_t (t^j - H^j) \exp[-\mathbf{A} \cdot (\mathbf{t}^p - \mathbf{h}^p)], \quad j = 1, \dots, m. \quad (92)$$

The partial derivative of (92) with respect to A_i yields the generic element of the Hessian matrix:

$$\frac{\partial^2 P_{\varepsilon;H}}{\partial A_i \partial A_j}(\mathbf{A}) = \sum_{t=1}^N c_t (t^j - H^j)(t^i - H^i) \exp[-\mathbf{A} \cdot (\mathbf{t}^p - \mathbf{h}^p)], \quad i = 1, \dots, m; j = 1, \dots, m. \quad (93)$$

To simplify notation, let

$$f_t \equiv c_t \exp[-\mathbf{A} \cdot (\mathbf{t}^p - \mathbf{h}^p)], \quad t = 1, \dots, N \quad (94)$$

denote the portfolio's cashflow received at time t expressed in future value (i.e. in horizon H 's value). Notice how each f_t depends upon $\mathbf{A}$.

Let Q_{ij} denote the generic element of the Hessian matrix $\partial^2 P_{\varepsilon, H}(\mathbf{a}) / \partial A_i \partial A_j$, $i = 1, \dots, m$; $j = 1, \dots, m$. Q_{ij} is then

$$\begin{aligned} Q_{i,j} &= \sum_{t=1}^N f_t (t^j - H^j) (t^i - H^i) = \sum_{t=1}^N f_t (t^i t^j - t^i H^j - H^i t^j + H^i H^j) \\ &= \sum_{t=1}^N f_t t^{i+j} - H^j \sum_{t=1}^N f_t t^i - H^i \sum_{t=1}^N f_t t^j + H^{i+j} \sum_{t=1}^N f_t, \\ &\quad i = 1, \dots, m; j = 1, \dots, m. \end{aligned} \quad (95)$$

From the first-order conditions (85)

$$\frac{1}{P_0} \sum_{t=1}^N t^j c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) = H^j, \quad j = 1, \dots, m \quad (85)$$

we can deduce

$$\sum_{t=1}^N t^j c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) = H^j \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) \quad (96)$$

and

$$\sum_{t=1}^N t^j c_t \exp[-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p)] = H^j \sum_{t=1}^N c_t \exp[-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p)].$$

Therefore, we have

$$\sum_{t=1}^N f_t t^j = H^j \sum_{t=1}^N f_t, \quad j = 1, \dots, m$$

which is also valid when j is replaced by i ($i = 1, \dots, m$). In expression (95), $\sum_{t=1}^N f_t t^i$ and $\sum_{t=1}^N f_t t^j$ can thus be replaced by $H^i \sum_{t=1}^N f_t$ and $H^j \sum_{t=1}^N f_t$ respectively, and the generic term of the Hessian at $\mathbf{a}$ becomes

$$Q_{i,j}(\mathbf{a}) = \sum_{t=1}^N f_t t^{i+j} - H^{i+j} \sum_{t=1}^N f_t \quad (97)$$

or, reverting to the notation using the c_t 's

$$Q_{i,j}(\mathbf{a}) = \sum_{t=1}^N c_t \exp[-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p)] t^{i+j} - H^{i+j} \sum_{t=1}^N c_t \exp[-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p)]. \quad (98)$$

Multiplying and dividing first term in the right-hand side of equation (98) by $P_0 = \sum_{t=1}^N c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p)$,

$$\begin{aligned}
Q_{i,j}(\mathbf{a}) &= P_0 \sum_{t=1}^N t^{i+j} c_t \exp[-\mathbf{a} \cdot (\mathbf{t}^p - \mathbf{h}^p)] / P_0 - H^{i+j} P_0 \exp(\mathbf{a} \cdot \mathbf{h}^p) \\
&= P_0 \exp(\mathbf{a} \cdot \mathbf{h}^p) \mu_{i+j}^{(P)} - H^{i+j} P_0 \exp(\mathbf{a} \cdot \mathbf{h}^p) \\
&= P_0 \exp(\mathbf{a} \cdot \mathbf{h}^p) [\mu_{i+j}^{(P)} - H^{i+j}] \\
&= P_{0,H} [\mu_{i+j}^{(P)} - H^{i+j}], \quad 1 \leq i \leq m, 1 \leq j \leq m.
\end{aligned} \tag{99}$$

Call the difference between the moment of order k and the k th power H the *generalized variance* of order k of the portfolio, and denote this generalized variance

$$\gamma_k = \mu_k^{(P)} - H^k, \quad k > 0. \tag{100}$$

All elements $Q_{i,j}(\mathbf{a})$ of the Hessian matrix are those of the matrix $\mu_{i+j}^{(P)} - H^{i+j} = \gamma_{i+j} \equiv \gamma_{i,j}$, each one being multiplied by the positive constant $P_{0,H}$. So the discussion of the positive semi-definiteness of matrix $\mathbf{Q}$ whose elements are $Q_{i,j}$ reduces to that of matrix $\mathbf{\Gamma}_m \equiv (\gamma_{i+j})$.

The first-order conditions imply that all generalized variances up to the m th order must vanish. So all elements in the triangle above the second principal diagonal¹² of the $\mathbf{\Gamma}_m$ matrix are equal to zero. Observe also that all elements in any diagonal parallel to the second diagonal must be equal to one another: indeed $\gamma_{i,j} = \gamma_{i+j}$.

We are now left with the task of looking for the properties that matrix $\mathbf{\Gamma}_m$ must exhibit in order to be positive semi-definite. $\mathbf{\Gamma}_m$ may be written as follows:

$$\mathbf{\Gamma}_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & \gamma_{m+1} \\ 0 & 0 & & \gamma_{m+1} & \gamma_{m+2} \\ & & & \gamma_{m+2} & \gamma_{m+3} \\ 0 & & \vdots & \vdots & \\ \vdots & \gamma_{m+1} & & & \\ \gamma_{m+1} & \gamma_{m+2} & \gamma_{m+3} & \cdots & \gamma_{2m} \end{bmatrix}.$$

For $\mathbf{\Gamma}_m$ to be positive semi-definite, all its principal minors must be either positive or zero. Consider, in the above matrix, the (2x2) principal minor exhibiting one zero in its upper-left corner. There are two possibilities:

¹²We call "second principal diagonal" of matrix $\mathbf{\Gamma}_m$ the diagonal between its lower-left element and its upper-right element (this diagonal has m elements, all equal to γ_{m+1}).

i) m is even; the above-mentioned principal minor is then of the form

$$\begin{vmatrix} 0 & \gamma_{m+1} \\ \gamma_{m+1} & \gamma_{m+2} \end{vmatrix}.$$

For this minor to be positive or zero, γ_{m+1} must be zero.

ii) m is odd; the principal minor is

$$\begin{vmatrix} 0 & 0 \\ 0 & \gamma_{m+1} \end{vmatrix}$$

and, adding one line and one column, an adjacent principal minor is then

$$\begin{vmatrix} 0 & 0 & \gamma_{m+1} \\ 0 & \gamma_{m+1} & \gamma_{m+2} \\ \gamma_{m+1} & \gamma_{m+2} & \gamma_{m+3} \end{vmatrix}$$

which must also be positive or zero. For Γ_m to be positive semi-definite, γ_{m+1} has to be positive or equal to zero. Suppose it is positive. Then this adjacent minor is equal to $-\gamma_{m+1}^3$, which is negative, entailing a contradiction. Thus γ_{m+1} must be equal to zero.

Let us now take up γ_{m+2} . Consider again case i). γ_{m+2} being a principal minor, it must be either positive or equal to zero. Suppose it is positive. An adjacent principal minor is

$$\begin{vmatrix} 0 & \gamma_{m+1} & \gamma_{m+2} \\ \gamma_{m+1} & \gamma_{m+2} & \gamma_{m+3} \\ \gamma_{m+2} & \gamma_{m+3} & \gamma_{m+4} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \gamma_{m+2} \\ 0 & \gamma_{m+2} & \gamma_{m+3} \\ \gamma_{m+2} & \gamma_{m+3} & \gamma_{m+4} \end{vmatrix}$$

(from what has been shown above), and its value is $-\gamma_{m+2}^3 < 0$, which is not feasible. So γ_{m+2} must be zero.

In the ii) case, a principal minor is

$$\begin{vmatrix} \gamma_{m+1} & \gamma_{m+2} \\ \gamma_{m+2} & \gamma_{m+3} \end{vmatrix} = \begin{vmatrix} 0 & \gamma_{m+2} \\ \gamma_{m+2} & \gamma_{m+3} \end{vmatrix},$$

and its value is $-\gamma_{m+2}^2$. For it to be positive or zero, γ_{m+2} has to be zero.

Carrying on the same argument, all terms γ_k must be equal to zero, except the last one, equal to $\gamma_{2m} = \mu_{2m} - H^{2m}$, which can be set at any (arbitrary) positive value K , expressed in $(\text{years})^{2m}$.

So the first-order conditions imply the m equations

$$\begin{aligned}\mu_1^{(P)} &= H \\ \mu_2^{(P)} &= H^2 \\ &\vdots \\ \mu_m^{(P)} &= H^m.\end{aligned}$$

The second-order conditions imply the m following equations

$$\begin{aligned}\mu_{m+1}^{(P)} &= H^{m+1} \\ \mu_{m+2}^{(P)} &= H^{m+2} \\ &\vdots \\ \mu_{2m-1}^{(P)} &= H^{2m-1} \\ \mu_{2m}^{(P)} &= H^{2m} + K\end{aligned}$$

to which finally the accounting constraint $\mu_0 = 1$ must be added, for a total of $2m+1$ equations, implying $2m+1$ bonds in the immunizing portfolio. This completes the proof of theorem 5. To illustrate, if a polynomial of order 3 is used for the spot rate structure, 4 coefficients correspond to that polynomial ($m = 4$). The immunizing portfolio will then be made out of $2m + 1 = 9$ bonds.

The quantities n_l ($l = 1, \dots, 2m + 1$) of each bond will be the solution of the following system of $2m + 1$ equations:

$$\begin{aligned}\frac{B_0^1(\mathbf{a})}{P_0(\mathbf{a})}\mu_0^{(1)}n_1 + \frac{B_0^2(\mathbf{a})}{P_0(\mathbf{a})}\mu_0^{(2)}n_2 + \dots + \frac{B_0^l(\mathbf{a})}{P_0(\mathbf{a})}\mu_0^{(l)}n_l + \dots + \frac{B_0^{(2m+1)}(\mathbf{a})}{P_0(\mathbf{a})}\mu_0^{(2m+1)}n_{2m+1} &= 1 \\ \frac{B_0^1(\mathbf{a})}{P_0(\mathbf{a})}\mu_1^{(1)}n_1 + \frac{B_0^2(\mathbf{a})}{P_0(\mathbf{a})}\mu_1^{(2)}n_2 + \dots + \frac{B_0^l(\mathbf{a})}{P_0(\mathbf{a})}\mu_1^{(l)}n_l + \dots + \frac{B_0^{(2m+1)}(\mathbf{a})}{P_0(\mathbf{a})}\mu_1^{(2m+1)}n_{2m+1} &= H \\ \dots &\dots \dots \dots \dots \\ \frac{B_0^1(\mathbf{a})}{P_0(\mathbf{a})}\mu_k^{(1)}n_1 + \frac{B_0^2(\mathbf{a})}{P_0(\mathbf{a})}\mu_k^{(2)}n_2 + \dots + \frac{B_0^l(\mathbf{a})}{P_0(\mathbf{a})}\mu_k^{(l)}n_l + \dots + \frac{B_0^{(2m+1)}(\mathbf{a})}{P_0(\mathbf{a})}\mu_k^{(2m+1)}n_{2m+1} &= H^k \\ \dots &\dots \dots \dots \dots \\ \frac{B_0^1(\mathbf{a})}{P_0(\mathbf{a})}\mu_{2m}^{(1)}n_1 + \frac{B_0^2(\mathbf{a})}{P_0(\mathbf{a})}\mu_{2m}^{(2)}n_2 + \dots + \frac{B_0^l(\mathbf{a})}{P_0(\mathbf{a})}\mu_{2m}^{(l)}n_l + \dots + \frac{B_0^{(2m+1)}(\mathbf{a})}{P_0(\mathbf{a})}\mu_{2m}^{(2m+1)}n_{2m+1} &= H^{2m} + K\end{aligned}\tag{101}$$

Let $\mathbf{M}$ denote the $(2m + 1) \times (2m + 1)$ matrix whose generic term is $[B_0^l(\mathbf{a})/P_0(\mathbf{a})]\mu_k^{(l)}$, $k = 0, \dots, 2m$; $l = 1, \dots, 2m + 1$. Each of those elements

is the k th moment of bond l weighted by the ratio of bond l 's initial value $B_0^l(\mathbf{a})$ to the portfolio's initial value $P_0(\mathbf{a})$. Let $\mathbf{n}$ be the $(2m+1)$ dimensional column vector $(n_1, \dots, n_l, \dots, n_{2m+1})$. Finally, let $\mathbf{h}_{2m+1}^p$ stand for the column vector $(1, H, \dots, H^k, \dots, H^{2m} + K)$. System (101) becomes¹³ simply

$$\mathbf{M}\mathbf{n} = \mathbf{h}_{2m+1}^p \quad (102)$$

which can be solved in n if and only if $\mathbf{M}$ is not singular. We get

$$\mathbf{n} = \mathbf{M}^{-1}\mathbf{h}_{2m+1}^p. \quad (103)$$

Let us now come back to the Hessian matrix. It is equal to:

$$P_{0,H}\Gamma = P_{0,H} \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ & & 0 & 0 \\ 0 & \dots & 0 & K \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ & & 0 & 0 \\ 0 & \dots & 0 & P_{0,H}K \end{bmatrix} \quad (104)$$

and therefore the associated quadratic form is

$$d\mathbf{A}^T \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ & & 0 & 0 \\ 0 & \dots & 0 & P_{0,H}K \end{bmatrix} d\mathbf{A} = P_{0,H}K dA_m^2. \quad (105)$$

Taking into account the fact that the gradient of P_H at $\mathbf{a}$, $\nabla P_H(\mathbf{a})$, is equal to zero, the Taylor development of $P_H(\mathbf{A})$ around point $\mathbf{a}$ yields

$$P_H(\mathbf{A}) - P_H(\mathbf{a}) = P_{0,H}(\mathbf{a})K dA_m^2 + R_2(\mathbf{a}, d\mathbf{A}) \quad (106)$$

where $R_2(\mathbf{a}, d\mathbf{A})$ denotes the remainder of the Taylor series. Since by definition $R_2(\mathbf{a}, d\mathbf{A})$ is finite, we can always determine a value of K such that $P_H(\mathbf{A}) > P_H(\mathbf{a})$, thus making $P_H(\mathbf{A})$ convex at point $\mathbf{a}$ whatever the magnitude of the displacement of $\mathbf{A}$, $d\mathbf{A}$. In applying our theorem, we will illustrate

¹³Notice that this is not the only way to envision system (101). An alternative would be to consider the shares $n_l B_0^l / P_0 \equiv \alpha_l$ ($l = 1, \dots, 2m+1$) and then the square matrix of $(2m+1)^2$ elements $\alpha_l \mu_k$ ($l = 1, \dots, 2m+1; k = 0, \dots, 2m$), each of those elements corresponding to line k and column l respectively. This matrix multiplies the column vector of the α_l 's, the unknowns. The system is thus solved in terms of the shares α_l instead of the numbers n_l . Both methods are of course equivalent in terms of results.

how simply increasing K , but increases the local convexity of the portfolio's future value.

We will now show that it is impossible to achieve immunization simply by meeting first order conditions. In other words, if one builds up a bond portfolio that conforms to first-order conditions only (all moments of order k , $k = 0, \dots, m$, are equal to the k th power of the horizon), it will never be possible to achieve second-order conditions for a minimum of the portfolio's future value. The likelihood that a portfolio formed in such a way that $\gamma_0, \gamma_1, \dots, \gamma_m = 0$ (such that it meets the first-order conditions) *and* that, in addition, it meets independtly *any* of the conditions $\gamma_{m+1}, \gamma_{m+2}, \dots, \gamma_{2m-1} = 0$, is nil. So is the probability that it meets all of them. Suppose then that one value of γ_k ($k = m + 1, \dots, 2m - 1$) differs ever so slightly from zero. Call this generalized variance γ . From what we have shown above, at least one line of elements parallel to the second diagonal of the Hessian matrix is made of γ 's. There are two possibilities: either γ is on the first diagonal, or it is not. We take them up in order:

a) γ is on the first diagonal. In that case γ must be positive. Also, there is a principal minor such as

$$\begin{vmatrix} 0 & 0 & \gamma \\ 0 & \gamma & \hat{\gamma} \\ \gamma & \hat{\gamma} & \gamma^* \end{vmatrix}$$

where $\hat{\gamma}$ and γ^* are as yet unspecified. The minor is equal to $-\gamma^3 < 0$; thus the portfolio cannot reach a minimum at a .

b) γ is not on the first diagonal. Then there is a principal minor such as

$$\begin{vmatrix} 0 & \gamma \\ \gamma & \hat{\gamma} \end{vmatrix}$$

which is equal to $-\gamma^2 < 0$ whatever the sign of γ , a contradiction with necessary conditions for a minimum.

3.8 The nature of the cash flows of an immunizing portfolio

Equations (101) look innocuous enough. However, they carry deep implications about the nature of the cash flows of the immunizing portfolio. Indeed, they imply that if an immunizing portfolio exists, *it must have one or more*

negative cash flows. Indeed, consider the second and the third equations of system (101). They stem from the first and the second equations of system (85), which can be written as

$$\sum_{t=1}^N t c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P(\mathbf{a}) = H \quad (85-1)$$

and

$$\sum_{t=1}^N t^2 c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P(\mathbf{a}) = H^2 \quad (85-2)$$

Both equations imply

$$\sum_{t=1}^N (t - H)^2 c_t \exp(-\mathbf{a} \cdot \mathbf{t}^p) / P(\mathbf{a}) = 0 \quad (85-3)$$

If equation (85-3) is to hold, as well as system (101), at least one cash flow c_t must be negative. In turn, this is possible if and only if at least one position n_l ($l = 1, \dots, L$) is negative. We will be able to verify, in the following applications, that immunizing portfolios *always* carry some negative cash flows, and therefore that immunizing portfolios have always at least one negative position.

4 Applications

Suppose we observe today a spot structure which, for reasons explained in Section 1, is represented by a third-order polynomial. The future spot rate curve, i.e. the structure that will prevail in, say, one year, is unknown and is assumed to remain unpredictable. We will now show with concrete examples how an immunizing portfolio is to be constructed, and we will put this portfolio to the test of very strong variations of the initial spot rate structure. In fact, those variations will even amount, in some cases, to changing the structure from a steeply increasing one to a decreasing one, and vice versa.

4.1 The spot structures and their shifts

We will consider eight arbitrary spot structures, labeled as $A, B, C, \dots, G, H$ ¹⁴. To each spot rate structure corresponds a triplet of points through which the spot rate curve passes. Those eight triplets are indicated in table 1.

Table 1. Basic spot rate structures, as defined by series of points $(t, s(t))$

Maturity t (years)	Spot rates $s(t)$ for various structures			
	$s_A(t)$	$s_B(t)$	$s_C(t)$	$s_D(t)$
	(per year, continuously compounded)			
0	0.04	0.06	0.048	0.04
5	0.05	0.06	0.054	0.047
$t = \bar{s} = 20$	0.06	0.06	0.06	0.055
Maturity t (years)	Spot rates $s(t)$ for various structures			
	$s_E(t)$	$s_F(t)$	$s_G(t)$	$s_H(t)$
	(per year, continuously compounded)			
0	0.053	0.035	0.06	0.049
5	0.060	0.042	0.063	0.053
$t = \bar{s} = 20$	0.066	0.052	0.065	0.06

¹⁴This designation is independent from that of the immunization horizon H .

Each of those curves is supposed to level off for maturity $t = \bar{s} = 20$ years. In order to find the coefficients of the corresponding polynomials, we have to solve system of equations (101). Doing so yields the following polynomials:

$$A : s_A(t) = 1.1(10^{-6})t^3 - 9.4(10^{-5})t^2 + 0.002444t + 0.04 \quad (107)$$

$$B : s_B(t) = 0.06$$

$$C : s_C(t) = 6.7(10^{-7})t^3 - 5.7(10^{-5})t^2 + 0.001467t + 0.048 \quad (108)$$

$$D : s_D(t) = 3.9(10^{-7})t^3 - 5.3(10^{-5})t^2 + 0.001656t + 0.04 \quad (109)$$

$$E : s_E(t) = 1.2(10^{-6})t^3 - 7.9(10^{-5})t^2 + 0.001767t + 0.053 \quad (110)$$

$$F : s_F(t) = -3.9(10^{-7})t^3 - 2.7(10^{-5})t^2 + 0.001544t + 0.035 \quad (111)$$

$$G : s_G(t) = -8.3(10^{-7})t^3 + 0.000051t^2 - 0.00103t + 0.052 \quad (112)$$

$$H : s_H(t) = -7.2(10^{-7})t^3 + 1.4(10^{-6})t^2 + 0.000811t + 0.049 \quad (113)$$

These eight structures are represented in Figures 6 and 7. They can be combined into $8 \times 7 = 56$ pairs: $AB, AC, \dots, HG$ each pair designating a scenario. The first letter corresponds to the initial structure; the second one designates the structure that is observed immediately *after* the bond portfolio's purchase. Any of those 56 scenarios can generate an infinity of sub-scenarios, in the following way.

Let $\lambda \in [-1, 2]$ denote a parameter (the definition interval of λ is arbitrary; it could be made larger or smaller). Let $s_a(t)$ be an initial spot rate structure; let $s_b(t)$ designate a new one. Consider the spot rate (denoted $s_{ab,\lambda}(t)$) defined by

$$s_{ab,\lambda}(t) = s_a(t) + \lambda[s_b(t) - s_a(t)] = (1 - \lambda)s_a(t) + \lambda s_b(t) \quad (114)$$

Thus $s_{ab,\lambda}(t)$ is a linear combination of the spot rates s_a and s_b ; particular cases correspond to $\lambda = 0$ ($s_{ab,0}(t) = s_a(t)$, the initial structure) and $\lambda = 1$ ($s_{ab,1}(t) = s_b(t)$, the "new" structure). Note that the initial spot structure s_a and the new spot rate s_b , together with λ , define a direction of change in the space of the polynomials' coefficients. This directional vector is simply $\lambda(\mathbf{A} - \mathbf{a})$ where $\mathbf{a}$ is the vector of the coefficients of the initial polynomial ($s_A(t)$ in this case) and $\mathbf{A}$ is the vector of the coefficients of the new polynomial ($s_B(t)$ here).

An infinity of variations of the spot rate curve can thus be generated by changing λ . Figure 8 illustrates such variations with λ taking the values -1; -0.5; 0; 0.5; 1; 1.5. The initial spot rate curve ($\lambda = 0$) is s_D : it is steeply

Spot rate

Figure 6. Spot structures $s_A(t)$, $s_B(t)$, $s_C(t)$ and $s_D(t)$
(continuously compounded yearly rates)

Spot rate

Figure 7. Spot structures $s_E(t)$, $s_F(t)$, $s_G(t)$ and $s_H(t)$
(continuously compounded yearly rates)

Spot rate

Figure 8. Varying parameter λ generates an infinity of spot rate curves.

increasing (from 4% for short rates to 5.5% for long ones). The new structure ($\lambda = -1$) is s_B , a horizontal at 6%. The values $\lambda = -1$ and $\lambda = -0.5$ correspond to a decrease of all rates; the value $\lambda = 0.5$ yields intermediate spot rates between s_D and s_B . As to $\lambda = 1.5$, it leads to an inversion of the structure, i.e. to a decreasing spot rate. Indeed, from the definition

$$s_{ab,\lambda}(t) = s_a(t) + \lambda[s_b(t) - s_a(t)]$$

we have

$$s'_{ab,\lambda}(t) = s'_a(t) + \lambda[s'_b(t) - s'_a(t)],$$

from which we deduce immediately the rule about the slope of $s_{ab,\lambda}(t)$:

- if $s'_b > s'_a$

$$s'_{ab,\lambda}(t) \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ if and only if } \lambda \begin{matrix} \geq \\ \leq \end{matrix} \frac{s'_a(t)}{s'_a(t) - s'_b(t)}$$

- if $s'_b < s'_a$

$$s'_{ab,\lambda}(t) \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ if and only if } \lambda \begin{matrix} \leq \\ \geq \end{matrix} \frac{s'_a(t)}{s'_a(t) - s'_b(t)}$$

- if $s'_b = s'_a$, $s'_{ab,\lambda}(t) = s'_a(t) = s'_b(t)$.

In one example, $s'_a(t) = s'_D(t) > 0$ and $s'_b(t) = s'_B(t) = 0$; so we are in the case $s'_b < s'_a$, and we have an inversion of the spot rate curve if and only if $\lambda > 1$.

We are now ready to describe the immunization process by which our investor can protect himself against *any* change in the structure of spot rates.

4.2 Building immunizing portfolios

Suppose that we have at our disposal 9 bonds, labeled $B_l, l = 1, \dots, 9$, whose characteristics are summarized in the following table:

Table 2. Basic characteristics of bonds used in immunization portfolio

Bond number	1	2	3	4	5	6	7	8	9
Coupon rate (in percent)	4	4.75	7	8	6	4.5	7	5	5.5
Maturity (years)	7	8	15	20	6	9	16	12	10

Suppose also that our initial spot rate curve is given by $s_A(t)$

$$s_A(t) = 1.1(10^{-6})t^3 - 9.4(10^{-5})t^2 + 0.002444t + 0.04$$

Recall that we have to determine the vector $\mathbf{n} = n_1, \dots, n_9$ that solves system

$$\mathbf{M}\mathbf{n} = \mathbf{h}^p.$$

In other words, $\mathbf{n}$ is given by

$$\mathbf{n} = \mathbf{M}^{-1}\mathbf{h}^p.$$

So our first task is to build matrix $\mathbf{M}$ whose elements are $[B_0^l(a)/p_0(a)]\mu_k^{(l)}$, $k = 0, \dots, 2m$; $l = 1, \dots, 2m + 1$. Using the spot structure $s_A(t)$, we obtain the values of the bonds and their various moments. The results are presented in table 3, which can be viewed as a (9x9) matrix. Recall that the moment of order k is expressed in (years) ^{k} .

Note that the (1x9) first line of matrix $\mathbf{M}$ is the first line of table 3 (the initial value of each bond expressed in terms of its par value); each subsequent line of matrix $\mathbf{M}$ is the corresponding line of table 3, scaled by the value of each bond in par value. Matrix $\mathbf{M}$ can thus be seen as table 3 premultiplied by the diagonal matrix of the values of the bonds.

Finally, suppose our immunization horizon is 7 years. We know that all generalized variances of order 0 to $2m - 1$ should be equal to zero, and

that the generalized variance of order m , γ_{2m} , should be positive. Since $\gamma_{2m} = \gamma_8 = \mu_8 - h^8$, the way to achieve this is to equate μ_8 to H^8 plus a positive constant K . Set this constant arbitrarily to 200,000 (expressed in (years)⁸). This amounts to setting K at a relatively low level, equal to about 3.4% of H^8 . We shall discuss further the consequences, for the immunizing process, of setting K at various levels. For the time being, we can rest assured that a local minimum will be attained for the portfolio's future value. Inverting matrix $\mathbf{M}$, $\mathbf{n}$ is simply :

$$\mathbf{n} = \mathbf{M}^{-1} \cdot \mathbf{h}$$

where $\mathbf{h}$ is given by:

$$\mathbf{h} = \begin{bmatrix} 1 \\ 7 \\ 49 \\ 343 \\ 2,401 \\ 16,807 \\ 117,649 \\ 823,543 \\ 5,964,801 \end{bmatrix}.$$

Performing the necessary calculations, we obtain the following immunizing portfolio:

$$\begin{array}{lll} n_1 = 6.161 & n_4 = 0.009 & n_7 = -0.448 \\ n_2 = -6.534 & n_5 = -1.211 & n_8 = 1.512 \\ n_3 = 0.823 & n_6 = 2.474 & n_9 = -1.371 \end{array}$$

4.3 Immunization results

We now have to check that this portfolio protects us against *any* change in the spot rate structure. To that effect we will consider all scenarios $AB, AC, \dots, HG$, and for each scenario a set of λ values from -1 to $+1.5$. Table 4 provides the results. The first figure in each cell indicates the portfolio's value just after the change from the initial structure to the new one; the second figure is the future portfolio's value based upon the new structure. Consider, for instance, the third cell of the first line, under $\lambda = 0.5$. It carries

two numbers: the first figure (97.5335) indicates the new present value of the portfolio at time ε , i.e. for a structure half-way between structures A and B , and thus equal to

$$s_{AB,0.5}(t) = s_A(t) + 0.5[s_B(t) - s_A(t)].$$

The second figure (144.7809) is the future value of the portfolio at horizon 7 years under the new structure $s_{AB,0.5}(t)$: it tells us that the portfolio is perfectly immunized since that is the future value of the portfolio before the spot structure change.

From the results laid out in Table 4, we conclude of course that immunization is obtained in each of the scenarios starting with spot rate structure $s_A(t)$. The same conclusion would be reached should the initial spot structure be $s_B(t), \dots, s_H(t)$. Table 5 indicates, for each of those initial structures, the future value protected, as well as the corresponding portfolio.

A remarkable, far from intuitive, conclusion emerges: the optimal portfolios are *little sensitive* to different initial (observed) structures. Even quite different initial spot curves such as $s_A(t)$, $s_E(t)$, or $s_G(t)$ are not conducive to substantial modifications in the immunizing portfolios. Their orders of magnitude remain the same (within boundaries of 3%), and of course the position signs remain the same.

This weak sensitivity of the portfolio to the initial spot structure implies that the cashflows generated by an asset are more important than the exact discounting factors in the evaluation process: in other words, what counts most is the cashflow profile of each bond. We will confirm this when we look for the optimal portfolio that would be built up from different sets of bonds. We will introduce variations first in the coupons, then in maturities, and finally in both parameters. Prior to that, we should discuss the question of the magnitude of the immunization parameter K .

Table 3. Value and moments of order $k = 1, \dots, 2m = 8$ of each bond under spot rate structure:

$$s_A(t) = 1.1(10^{-6})t^3 - 9.4(10^{-5})t^2 + 0.002444t + 0.04$$

Bond	B_1	B_2	B_3	B_4	B_5
Initial value ¹⁵	0.9215	0.9537	1.1085	1.2343	1.0390
Moments					
μ_1	6.1940	6.7878	9.8975	11.3019	5.2211
μ_2	41.300	50.853	124.486	175.241	29.569
μ_3	282.14	393.60	1703.4	3058.9	172.29
μ_4	1946.9	3087.8	24174	56300	1016.0
μ_5	13501	24388	349522	1.065209	6028
μ_6	93,898	193,361	5,108,123	20,479,893	35,884
μ_7	654,177	1,536,761	75,155,699	4.0(10 ⁸)	214,086
μ_8	4,562,925	12,232,997	1.1(10 ⁹)	7.8(10 ⁹)	1,279,087
Bond	B_6	B_7	B_8	B_9	
Initial value	0.9250	1.1099	0.9505	0.9755	
Moments					
μ_1	7.5036	10.2761	8.0123	8.9826	
μ_2	62.849	136.15	73.349	95.892	
μ_3	545.57	1973.91	701.47	1084.2	
μ_4	4806.3	29757	6831.2	12556	
μ_5	42655	457680	67134	147197	
μ_6	380,145	7,120,937	663,180	1,737,533	
μ_7	3,396,730	1.1(10 ⁸)	6,572,335	20,598,422	
μ_8	30,403,315	1.8(10 ⁹)	65,273,038	2,4(10 ⁸)	

¹⁵This initial value is expressed in terms of its par value B_T . It is equal to B_0/B_T . Moments of order k are expressed in (years ^{k}).

Table 4. Immunization results for initial spot rate $s_A(t)$.
Present value of portfolio at time 0: 100. Future value protected
at time 0: 144.7809⁽¹⁾; $K = 200,000$

- first figure in each cell: present value of portfolio at time ε , under new spot structure
- second figure: future value of portfolio under at time ε , under new spot structure.

scenario	$\lambda = -1$	$\lambda = -0.5$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$
AB	105.1217	102.5289	97.5335	95.1278	92.7815
	144.7809	144.7809	144.7809	144.7809	144.7809
AC	102.0181	101.044	99.006	98.0219	97.0475
	144.7809	144.7809	144.7809	144.7809	144.7809
AD	97.4147	98.6989	101.3183	102.6539	104.0072
	144.7809	144.7809	144.7809	144.7809	144.7809
AE	106.52	103.2085	96.8912	93.8791	90.9606
	144.7809	144.7809	144.7809	144.7809	144.7809
AF	94.2190	97.0664	103.0222	106.1358	109.3435
	144.7809	144.7809	144.7809	144.7809	144.7809
AG	95.9591	97.0664	102.0839	104.2112	106.3829
	144.7809	144.7809	144.7809	144.7809	144.7809
AH	101.1502	100.5734	99.4299	98.8630	98.2994
	144.7809	144.7809	144.7809	144.7809	144.7809

⁽¹⁾ This value is equal to $100 \exp[s_A(7) \cdot 7]$.

Table 5. Future value protected and immunizing portfolio for each initial spot rate structure $s_A(t), \dots, s_H(t)$ Horizon $H = 7$ years; $K = 200,000$.

Initial structure	Future value protected	Immunizing portfolio					
		n_1 n_5	n_2 n_6	n_3 n_7	n_4 n_8	n_9	
$s_A(t)$	144.7809	6.161	-6.534	0.823	0.009		
		-1.211	2.474	-0.448	1.512	-1.371	
$s_B(t)$	152.1962	6.475	-6.857	0.833	0.009		
		-1.273	2.598	-0.452	1.562	-1.407	
$s_C(t)$	147.7026	6.285	-6.662	0.827	0.009		
		-1.236	2.523	-0.450	1.532	-1.386	
$s_D(t)$	141.0378	5.993	-6.306	0.765	0.008		
		-1.186	2.355	-0.414	1.460	-1.297	
$s_E(t)$	154.2206	6.5724	-7.0296	0.9055	0.0098		
		-1.2828	2.7049	-0.4953	1.6101	-1.4893	
$s_F(t)$	136.411	5.7910	-6.0620	0.7273	0.0074		
		-1.1510	2.2419	-0.3935	1.4103	-1.2402	
$s_G(t)$	138.9303	5.8877	-6.0986	0.6679	0.0064		
		-1.1782	2.2214	-0.3566	1.3961	-1.1890	
$s_H(t)$	146.446	6.2298	-6.5922	0.8216	0.0087		
		-1.2272	2.4904	-0.4476	1.5173	-1.3713	

4.4 How large should we set the immunization parameter K ?

Recall that we introduced parameter K in order to set the generalized variance $\gamma_{2m+1} = \gamma$ at a positive number. We had set arbitrarily K at 200,000. For that value of K , it is always possible to protect the investor against any change generated in any possible scenario $AB, \dots, HG$ for values of λ between -1 and $+1.5$. This means that immunization is warranted *at least* in this interval. We will now study this relationship.

We have to keep in mind that the immunization process has the happy effect of making the future value $F_H(\lambda)$ an extremely flat curve in F_H, λ space,

and we have to ask the question whether the fact that the portfolio's future value is guaranteed within 4 decimal places (as table 4 reveals) warrants that this future value is *strictly* higher than the value to be protected. This prods us to examine each figure with more than 4 decimals. The conclusion is that for all scenarios these future values are indeed strictly larger than the aim, with one exception: if the initial structure is D and counter-pivots with respect to structure B , this strict inequality does not hold for values of λ such that $\lambda < -0.346$ – see Table 6. Of course, practitioners would be happy to protect an investment as well as in case $\lambda = -1$, for instance: Table 6 tells us that the future value of an initial \$100,000,000 portfolio is immunized against a very large, non-parallel shift of the spot curve within a margin smaller than \$4. Nevertheless, we want to study this case on the following grounds: it is important to be able to picture $B_H(\lambda)$ as a *locally convex* curve of λ , with a local minimum at λ , and to see how this curve is modified when changing K . From what we have shown previously, K should play the role of a local convexity parameter: the higher K , the more convex should be the curve in F_H, λ space, and our aim now is to verify this.

We shall first consider a close-up of the situation within a close vicinity of $\lambda = 0$, i.e. in the interval $\lambda \in [-0.2, +0.2]$, with the following immunization constants: $K = 200,000$; $K = 100,000$ and $K = 50,000$. The results conform exactly our predictions: a higher value of K enhances convexity of the $P_H(\lambda)$ curve in a vicinity of the minimum point $(0, P_H(0))$ as evidenced in figure 9.

Now let us pay attention to this family of curves. Observe that for positive values of λ , the curves exhibit strong convexity, whilst for negative values convexity is somewhat weaker, transforming relatively quickly into concavity. This leads us to surmise that, if complete immunization is easily achieved for positive values of λ , the portfolio's future value may well decline and even become slightly – smaller than the target is λ is sufficiently large in absolute value. This indeed turns out to be the case as evidenced by Figure 10 where we have considered values of λ from -0.6 to 0.24. If λ is sufficiently small, the future value starts declining and ultimately becomes lower – if ever slightly than $F_H(a)$. Let us remember, however, the orders of magnitude: in our example, the future value (at horizon 7 years) of \$100,000,000 today remains immunized within \$4! One striking feature of Figures 9-11 is to throw light onto the fact that K is a local convexity parameter: its sole predictable effect to increase convexity at the initial point a (equivalently: for $\lambda = 0$). (Observe how, around $\lambda = -0.36$, the curves interests).

Figure 9. Increasing K enhances local convexity of the portfolio's future value.

Figure 10. While increasing local convexity, higher values of K do not necessarily widen immunization interval.

Figure 11. At a wider scale, F_H may look going through an inflection point at $\lambda = 0$, although it undergoes a local minimum there.

We conclude from this analysis, and the one that can be carried out from a wider range of initial spot structures, that it is enough to use an immunization constant K in the order of 3% of the last moment.

Table 6. Results of immunization; scenario DB ;

horizon $H = 7$ years; $K = 200,000$

Target: $B_{0,H} = 141,037,819.08 = 100 \exp[7s_D(7)]$

λ	$B_H\{s_D(t) + \lambda[s_B(t) - s_D(\lambda)]\}$	
1	141,037,827.60	values higher or equal to target
0.8	141,037,823.70	
0.6	141,037,821.22	
0.4	141,037,819.83	
0.2	141,037,819.22	
0	141,037,819.08	
-0.2	141,037,819.12	
-0.346*	141,037,819.03	values lower than target
-0.4	141,037,819.03	
-0.6	141,037,818.53	
-0.8	141,037,817.33	
-1	141,037,815.15	

* Rounded value; $\lambda = -0.34596239916$.

4.5 Infinity of solutions

We are now ready to state our last theorem, concerning the number of solutions that correspond to the various possible values of K .

• Theorem 6

Suppose that the spot rate structure can be expanded into a Taylor series. Then there exist an infinity of immunizing bond portfolios.

The proof of this theorem rests upon the continuity of the immunization constant K .

The fact that an infinity of bond portfolios can protect the investor can have important practical applications. As we had indicated before, the outcome of the immunization process is to make the future value of the portfolio an extremely “flat” curve in (P_H, λ) space. Therefore the interest of having large choices for K will not be to try to enhance convexity; it will rather reside in the extended available choices when building portfolios.

A natural way to apply this important property could be to choose K such as to minimize the transaction costs of building up, or adjusting the bond portfolio.

4.6 How sensitive are immunizing portfolios to changes in horizon H ?

We now address the all-important question of the sensitivity of immunizing portfolios to changes in then horizon. In order to answer that question, we have chosen to study optimal portfolios corresponding to the initial spot rate structures $s_A(t)$, $s_B(t)$ and $s_G(t)$, for horizons ranging from three years to twenty years. This span of years for horizon H could have been wider, because theory tells us that immunizing portfolios do exist for a broader range of horizons. However, since the basket of bonds to chose from is the initial one, with maturities between 6 and 20 years, we may surmise that overly large negative and positive positions would be the rule if we attempted to perform immunization for very long or very short horizons with bonds to be chosen from such a basket. We will be able to confirm this.

For the time being, let us examine the results we get when immunizing under the initial spot rate structure $s_A(t)$. The portfolios for each horizon are indicated in table 7, and their evolution with respect to the horizon are pictured in figure 12 (bonds 1-5) and in figure 13 (bonds 6-9). It is quite striking to notice how stable the portfolios remain when the horizons are in the 6-16 years range. When horizons exceed that range, the portfolios become highly unstable.

Our initial structure ($s_A(t)$) was an increasing one. We can now confirm these results in the case of a constant structure ($s_B(t)$) and a decreasing one ($s_G(t)$) (see tables 8 and 9, and the corresponding figures 14 to 17). The similarity of the portfolios for any initial spot rate is striking. It generalizes the property we had shown previously for all spot structures from $s_A(t)$ to $s_G(t)$ when the horizon was 7 years. Now we can see that this property

applies to all immunization horizons. Furthermore, we can observe that for any kind of initial spot rate, portfolios are very stable for mid-range horizons, and become highly unstable outside that range.

Table 7. Immunizing portfolios according to horizon H , under initial spot rate structure $s_A(t)$; $K = 200,000$; number of each bonds in portfolio

Horizon H (years)	Immunizing portfolios								
	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
	4%	4.75%	7%	8%	6%	4.5%	7%	5%	5.5%
	7 yrs	8 yrs	15 yrs	20 yrs	6 yrs	9 yrs	10 yrs	12 yrs	10 yrs
3	-26.34	-26.98	-7.60	-0.05	15.93	103.35	3.65	-82.53	22.29
4	-8.69	-28.47	-3.47	-0.02	9.46	67.76	1.62	-47.72	11.38
5	2.45	-18.19	-0.40	0.001	3.24	26.81	0.13	-14.96	2.50
6	6.25	-9.92	0.71	0.008	-0.11	6.52	-0.40	-0.67	-0.96
7	6.16	-6.53	0.82	0.009	-1.21	2.47	-0.45	1.51	-1.37
8	5.36	-6.56	0.77	0.008	-1.33	4.26	-0.43	0.55	-1.19
9	5.23	-7.80	0.78	0.008	-1.30	5.48	-0.43	0.74	-1.22
10	5.70	-8.90	0.75	0.008	-1.39	5.13	-0.41	1.71	-1.09
11	6.13	-9.56	0.64	0.008	-1.49	5.04	-0.37	1.40	-0.30
12	6.21	-10.14	0.71	0.008	-1.48	6.64	-0.40	-0.95	0.91
13	6.22	-11.04	1.30	0.01	-1.43	9.27	-0.60	-3.79	1.56
14	6.69	-12.24	2.32	0.01	-1.51	10.76	-0.80	-4.52	0.75
15	7.57	-13.20	2.71	0.01	-1.74	10.03	-0.30	-2.64	-1.05
16	7.55	-13.15	0.29	0.01	-1.73	9.97	2.12	-2.61	-1.06
17	4.24	-11.75	-7.22	0.06	-0.72	17.87	7.47	-13.41	5.03
18	-3.50	-9.80	-19.06	0.3	1.76	39.06	14.73	-40.30	18.80
19	-9.58	-9.73	-25.21	1.04	3.84	58.68	17.15	-63.20	29.24
20	10.74	-15.28	4.50	2.97	-2.67	4.70	-3.83	5.35	-5.52

Figure 12. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_A ; bonds 1-5.

Figure 13. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_A ; bonds 6-9.

Table 8. Immunizing portfolios according to horizon H , under initial spot rate structure $s_B(t)$; $K = 200,000$; number of each bonds in portfolio

Horizon H (years)	Immunizing portfolios								
	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
	4%	4.75%	7%	8%	6%	4.5%	7%	5%	5.5%
	7 yrs	8 yrs	15 yrs	20 yrs	6 yrs	9 yrs	10 yrs	12 yrs	10 yrs
3	-27.46	-29.21	-7.76	-0.05	16.78	109.35	3.71	-86.68	23.12
4	-9.03	-30.23	-3.54	-0.02	9.95	71.31	1.64	-49.93	11.77
5	2.62	-19.19	-0.41	0.001	3.40	28.17	0.17	-15.65	2.59
6	6.58	-10.43	0.72	0.008	-0.12	6.85	-0.40	-0.72	-0.98
7	6.48	-6.86	0.83	0.009	-1.27	2.60	-0.45	1.56	-1.41
8	5.62	-6.87	0.78	0.008	-1.40	4.45	-0.43	0.57	-1.22
9	5.47	-8.14	0.79	0.008	-1.37	5.68	-0.43	0.79	-1.26
10	5.94	-9.24	0.76	0.008	-1.45	5.25	-0.42	1.83	-1.13
11	6.37	-9.87	0.65	0.008	-1.55	5.08	-0.37	1.55	-0.33
12	6.43	-10.40	0.72	0.008	-1.54	6.67	-0.40	-0.85	0.92
13	6.41	-11.26	1.33	0.01	-1.49	9.32	-0.61	-3.75	1.57
14	6.85	-12.43	2.37	0.01	-1.56	10.77	-0.82	-4.45	0.72
15	7.74	-13.33	2.76	0.01	-1.79	9.89	-0.31	-2.43	-1.14
16	7.69	-13.23	0.31	0.01	-1.78	9.76	2.13	-2.36	-1.15
17	4.22	-11.79	-7.29	0.06	-0.71	18.04	7.53	-13.60	5.12
18	-3.85	-9.87	-19.28	0.29	1.88	40.31	14.84	-41.63	18.80
19	-10.20	-9.87	-25.51	1.04	4.06	60.89	17.29	-63.20	19.29
20	10.94	-15.22	4.58	2.97	-2.74	4.99	-3.86	6.04	-5.75

Figure 14. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_B ; bonds 1-5.

Figure 15. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_B ; bonds 6-9.

Table 9. Immunizing portfolios according to horizon H , under initial spot rate structure $s_G(t)$; $K = 200,000$; number of each bonds in portfolio

Horizon H (years)	Immunizing portfolios								
	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
	4%	4.75%	7%	8%	6%	4.5%	7%	5%	5.5%
	7 yrs	8 yrs	15 yrs	20 yrs	6 yrs	9 yrs	10 yrs	12 yrs	10 yrs
3	-27.86	-29.91	-7.82	-0.05	17.06	111.40	3.73	-88.14	23.43
4	-9.16	-30.78	-3.56	-0.02	10.11	72.51	1.65	-50.71	11.92
5	2.66	-19.51	-0.42	0.001	3.46	28.62	0.14	-15.89	2.62
6	6.68	-10.6	0.72	0.008	-0.12	6.96	-0.40	-0.73	-0.99
7	6.58	-6.97	0.84	0.009	-1.30	2.64	-0.45	1.58	-1.42
8	5.71	-6.98	0.79	0.008	-1.42	4.52	-0.43	0.58	-1.23
9	5.56	-8.26	0.80	0.008	-1.39	5.76	-0.44	0.80	-1.28
10	6.03	-9.37	0.76	0.008	-1.47	5.32	-0.42	1.86	-1.14
11	6.46	-10.00	0.65	0.008	-1.57	5.13	-0.37	1.59	-0.33
12	6.51	-10.52	0.73	0.008	-1.56	6.72	-0.40	-0.83	0.92
13	6.48	-11.37	1.33	0.01	-1.50	9.38	-0.61	-3.75	1.58
14	6.92	-12.52	2.38	0.01	-1.57	10.81	-0.82	-4.44	0.72
15	7.80	-13.40	2.78	0.007	-1.80	9.87	-0.31	-2.38	-1.16
16	7.75	-13.27	0.32	0.007	-1.79	9.72	2.13	-2.30	-1.17
17	4.22	-11.81	-7.32	0.06	-0.71	18.11	7.55	-13.69	5.16
18	-3.98	-9.87	-19.38	0.3	1.92	40.75	14.90	-42.12	19.50
19	-10.42	-9.88	-25.64	1.04	4.13	61.65	17.36	-66.30	30.35
20	11.01	-15.21	4.61	2.97	-2.76	3.76	-3.88	6.28	-5.83

4.7 How sensitive are immunizing portfolios to a change in the basket of available bonds?

We will now show that the slightest modification in the composition of the basket of available bonds has major consequences on the immunizing portfolio.

Figure 16. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_G ; bonds 1-5.

Figure 17. Immunizing portfolios as a function of horizon H ;
initial spot structure: s_G ; bonds 6-9.

We revert to our base case: immunization with an initial spot rate $s_A(t)$; a horizon $H = 7$ years and a constant $K = 200,000$. We will consider an innocuous-looking, very small change in the features of two among the nine bonds that were available for our initial application, as described in Section 4.2, table 2. Let us keep all last seven bonds identical in their maturity and coupon. Let us keep also the maturities of bonds 1 and 2 identical (7 and 8 years, respectively, and let us simply interchange their coupons: the coupon of bond 1 becomes 4.75 and the coupon of bond 2 becomes 4). The results are quite surprising: as table 10 reveals, the immunizing portfolio undergoes huge shifts: some positions change sign: that is the case of bonds 1, 2, 5, 6. Other positions undergo huge shifts: n_8 , for instance, increases from 1.5 to 26.3.

Table 10. The sensitivity of immunizing portfolios to changes in coupon rates

Initial bond basket changes(*)	Immunizing portfolios								
	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
$c_1 = 4$ $c_2 = 4.75$ (Initial basket)	6.16	-6.53	0.82	0.009	-1.21	2.47	-0.45	1.51	-1.37
$c_1 = 4.75$	-10.33	32.12	2.13	0.016	1.77	-43.5	-1.05	26.26	-6.08
$c_2 = 4$ $c_6 = 5.5$ $c_9 = 4.5$	-1.55	11.57	1.43	0.01	0.19	-18.84	-0.73	13.01	-3.62

(*) In this column we indicate changes in coupons of some of the initial bonds.

We can do the same exercises with bonds 6 and 9: interchanging their coupons while keeping the same maturities leads to the portfolio described

in the third line of table 10. (Notice that in the cases considered above, interchanging coupons is, tantamount to exchanging maturities) as the reader can verify from table 2, Section 4.2. As to the immunizing results, it is of no surprise that they are as good as those corresponding to the initial basket.

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* *

5 By way of conclusion: a few suggestions

This study brought its fair share of (nice) surprises. First, a portfolio immunized against *any* change of the spot structure carries *always* at least one negative position. Also surprising is the simplicity of the central result: if an investor is protecting herself from any shift of a spot curve represented by an $m - 1$ order polynomial, she just has to build a portfolio such that its generalized variances of order 1 to $2m - 1$ are equal to zero, and its generalized variance of order $2m$ is equal to a positive, arbitrary number – which leads to the next shocker: even with a given set of $2m + 1$ available bonds, there exists an infinity of immunizing portfolios for one given, initial, spot rate curve.

Equally unforeseen is the fact that the structure of an immunizing portfolio depends so little on the initial, observed structure, or on the investor's objective (her horizon), and so much on the *nature* of the bonds used in the portfolio. Changing if ever so marginally the characteristics of one or two bonds in the basket of the available bonds may alter dramatically the composition of the whole bond portfolio.

These results prompt us to suggest that the next step, for the practitioner, would be to minimize transaction costs by making use of the fact that, even for a well-defined, limited, set of $2m + 1$ bonds, an infinity of immunizing portfolios are available. Minimizing transaction costs could be achieved, for instance, by studying numerically the relationship between the immunization parameter K and the transaction cost. Indeed, starting from an initial, observed, spot rate structure, to each value of the arbitrary, positive, parameter K corresponds a given immunizing portfolio, to which corresponds a given transaction cost. This could be done when building up a portfolio from scratch. But it could be done also in order to transform an existing, non-protected, portfolio into an immunized one at minimal cost.

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