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Abstract:

Barnett (1979), extending Theil's (1971) work, aggregates differential demand equations in absolute (undeflated) prices. This paper extends further these results by deriving aggregate (macro) demand equations in relative prices under stated assumptions. These are exactly analogous to the individual consumer's (micro) demand equations and all the properties of the microcoefficients carryover.

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1. Introduction

Theil (1971) aggregates differential demand equations in relative prices under fairly strong assumptions about the macro parameters and variables. Barnett (1979) relaxes these assumptions but works with absolute (undeflated) prices. In this paper we extend Theil's work under the weaker assumptions of Barnett and derive macro demand equations in relative prices. The results show that the properties of the macrocoefficients are completely analogous to those of the microcoefficients. In other words, there is no aggregation problem with these sorts of demand equations (at least under the stated assumptions).

2. Micro Demand Equations in Relative Prices

The relative price version of the differential demand equation for good i and consumer c can be written as (Theil, 1971, page 581)

$$(1) \quad w_{ic} d(\log q_{ic}) = \theta_{ic} d(\log Q_c) + \sum_{j=1}^n v_{ijc} \left[d(\log p_j) - \sum_{k=1}^n \theta_{kc} d(\log p_k) \right],$$

where $w_{ic} = p_i q_{ic} / M_c$ is the budget share of good i for consumer c ; q_{ic} is the quantity consumed of i by c ; p_i is the price of i ; M_c is the income of c ; θ_{ic} is the marginal share of i ; $d(\log Q_c) = d(\log M_c) - \sum_{k=1}^n w_{kc} d(\log p_k)$ is the change in the real income of c . The microcoefficients θ_{ic} and v_{ijc} of the micro demand equation (1) are functions of income and prices and

satisfy

$$(2) \quad \sum_{i=1}^n \theta_{ic} = 1, \quad \sum_{j=1}^n v_{ijc} = \phi_c \theta_{ic}, \quad i=1, \dots, n,$$

where ϕ_c is income flexibility, while the matrix $[v_{ijc}]$ is symmetric negative definite for each c .

Let there be N consumers. We define the per capita variables which we shall refer to as the macrovariables

$$(3) \quad \bar{q}_i = \frac{1}{N} \sum_{c=1}^N q_{ic}, \quad \bar{M} = \frac{1}{N} \sum_{c=1}^N M_c = \sum_{i=1}^n p_i \bar{q}_i, \quad \bar{w}_i = \frac{p_i \bar{q}_i}{\bar{M}}, \quad i=1, \dots, n.$$

Our objective is to develop a macro demand equation in terms of macro variables and macrocoefficients which is analogous to the micro demand equation (1).

Let $K_{\ell c} = M_c (\theta_{\ell c} - \bar{\theta}_{\ell})$. Following Barnett (1979), we assume that M_c , $c=1, \dots, N$, are identically independently distributed and that the moments $E[M_c]$, $E[M_c \theta_{ic}]$, $E[M_c v_{ijc}]$ and $E[K_{\ell c} v_{ijc}]$ exist and are finite. We also define the macrocoefficients as

$$(4) \quad \bar{\theta}_i = \frac{E[M_c \theta_{ic}]}{E[M_c]}, \quad \bar{v}_{ij} = \frac{E[M_c v_{ijc}]}{E[M_c]}, \quad i, j=1, \dots, n, \quad \bar{\phi} = E[\phi_c],$$

and assume that ϕ_c and $M_c \theta_{ic}$ are stochastically independent. The macrocoefficients $\bar{\theta}_i$ and $\bar{v}_{ij}$ are population versions of weighted average microcoefficients, with weights proportional to corresponding incomes.

3. Aggregation Over Consumers

To obtain the macro demand equation we multiply both sides of (1) by $M_c/\bar{M}$ and sum over $c=1, \dots, N$ to yield

$$(5) \quad \sum_{c=1}^N \frac{M_c}{\bar{M}} w_{ic} d(\log q_{ic}) = \sum_{c=1}^N \frac{M_c}{\bar{M}} \theta_{ic} d(\log Q_c) + \frac{1}{\bar{M}} \sum_{j=1}^n \left[\frac{1}{N} \sum_{c=1}^N M_c v_{ijc} \right] d(\log p_j) \\ - \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{k=1}^n \frac{1}{N} \sum_{c=1}^N v_{ijc} \theta_{kc} M_c d(\log p_k).$$

We can easily show that the left-hand term of equation (5) can be simplified to $\bar{w}_i d(\log \bar{q}_i)$, which is the macro analogue of the left-hand term of (1). For sufficiently large N and under the above assumption, Barnett (1979) employs Khinchine and Slutsky theorems to show that the real income term on the right-hand side of equation (5) can be simplified to $\bar{\theta}_i d(\log \bar{Q})$, where $d(\log \bar{Q}) = d(\log \bar{M}) - \sum_{k=1}^n \bar{w}_k d(\log p_k)$ is the change in macro real income.

As M_c and $M_c v_{ijc}$, $c=1, \dots, N$ are i.i.d., applying Khinchine's theorem gives

$$\bar{M} = E[M_c] + o_p(1) \quad \text{and} \quad \frac{1}{N} \sum_{c=1}^N M_c v_{ijc} = E[M_c v_{ijc}] + o_p(1).$$

where $o_p(1)$ denotes a random variable that converges in probability to zero as N tends to infinity. Assuming $E[M_c] > 0$, applying Slutsky's theorem, we get

$$\frac{1}{\bar{M}} = \frac{1}{E[M_c]} + o_p(1).$$

Therefore, for sufficiently large N , the second term on the right-hand side of (5) becomes

$$(6) \quad \lim_{N \rightarrow \infty} \frac{1}{\bar{M}} \sum_{j=1}^n \left[\frac{1}{N} \sum_{c=1}^N M_c v_{ijc} \right] d(\log p_j) = \sum_{j=1}^n \bar{v}_{ij} d(\log p_j),$$

where we have also used (4) and $\text{plim}_{N \rightarrow \infty} o_p(1) = 0$.

We write the third right-hand term of (5) as

$$(7) \quad \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{k=1}^n \frac{1}{N} \sum_{c=1}^N v_{ijc} \theta_{kc} M_c d(\log p_k) = \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{k=1}^n \left[\frac{1}{N} \sum_{c=1}^N M_c v_{ijc} \right] \bar{\theta}_k d(\log p_k) \\ + \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{\ell=1}^n \frac{1}{N} \sum_{c=1}^N M_c (\theta_{\ell c} - \bar{\theta}_\ell) v_{ijc} d(\log p_\ell).$$

Using (6), for sufficiently large N , the first term on the right-hand side of equation (7) becomes

$$(8) \quad \lim_{N \rightarrow \infty} \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{k=1}^n \left[\frac{1}{N} \sum_{c=1}^N M_c v_{ijc} \right] \bar{\theta}_k d(\log p_k) = \sum_{j=1}^n \bar{v}_{ij} \sum_{k=1}^n \bar{\theta}_k d(\log p_k).$$

As $K_{\ell c} = M_c (\theta_{\ell c} - \bar{\theta}_\ell)$ and using (4) we obtain $E[K_{\ell c}] = 0$. Thus, the second term on the right-hand side of (7) can be simplified to

$$\frac{1}{\bar{M}} \sum_{j=1}^n \sum_{\ell=1}^n \left[\frac{1}{N} \sum_{c=1}^N K_{\ell c} v_{ijc} \right] d(\log p_\ell) = \sum_{j=1}^n \sum_{\ell=1}^n \frac{\text{cov}[K_{\ell c}, v_{ijc}]}{E[M_c]} d(\log p_\ell) + o_p(1),$$

where we have again used Khinchine's theorem to obtain $(1/N) \sum_{c=1}^N K_{\ell c} v_{ijc} = E[K_{\ell c} v_{ijc}] + o_p(1)$ and the result that $1/\bar{M} = 1/E[M_c] + o_p(1)$. In the aggregation of the absolute price version of the demand equation, Barnett (1979) also obtained a similar ratio $\text{cov}[K_{\ell c}, V_c]/E[M_c]$, where

$V_c = d(\log Q_c)$. He then uses Schwartz' inequality to obtain an upper bound for this ratio, $\{\text{var}[K_{\ell c}] \text{var}[V_c]\}^{1/2}/E[M_c]$. For a developed economy, he argues that, as the variability that induces randomness into $(K_{\ell c}, V_c)$ is likely to be small relative to $E[M_c]$, the ratio $\text{cov}[K_{\ell c}, V_c]/E[M_c]$ is typically small. Using the same argument, we also conclude that the ratio $\text{cov}[K_{\ell c}, v_{ijc}]/E[M_c]$ will also be small. Therefore,

$$(9) \quad \lim_{N \rightarrow \infty} \frac{1}{\bar{M}} \sum_{j=1}^n \sum_{\ell=1}^n \frac{1}{N} \sum_{c=1}^N M_c (\theta_{\ell c} - \bar{\theta}_{\ell}) v_{ijc} d(\log p_{\ell}) = 0.$$

Hence, it follows from equations (6)-(9) that, for sufficiently large N , we can simplify (5) as

$$\bar{w}_i d(\log \bar{q}_i) = \bar{\theta}_i d(\log \bar{Q}) + \sum_{j=1}^n \bar{v}_{ij} \left[d(\log p_j) - \sum_{k=1}^n \bar{\theta}_k d(\log p_k) \right],$$

$i=1, \dots, n.$

which is the macro analogue of (1).

4. Properties of the Macrocoefficients

If we multiply both sides of the equations in (2) by $M_c/E[M_c]$, take the mathematical expectation of both sides and use (4) we obtain

$$\sum_{i=1}^n \bar{\theta}_i = 1, \quad \sum_{j=1}^n \bar{v}_{ij} = \bar{\phi} \bar{\theta}_i.$$

These are the macro versions of the equations in (2).

Now we shall show that the macro price coefficient matrix $[\bar{v}_{ij}]$ is symmetric negative definite. As the micro price coefficients v_{ijc} 's are symmetric, from the definition of $\bar{v}_{ij}$, it follows that the $\bar{v}_{ij}$'s are also symmetric. To prove negative definiteness, consider the quadratic form

$$\sum_{i=1}^n \sum_{j=1}^n x_i \bar{v}_{ij} x_j = \frac{1}{E[M_c]} \sum_{i=1}^n \sum_{j=1}^n x_i E[M_c v_{ijc}] x_j = \frac{1}{E[M_c]} E \left[M_c \sum_{i=1}^n \sum_{j=1}^n x_i v_{ijc} x_j \right] < 0,$$

where we have used the fact that the matrices $[v_{ijc}]$ are negative definite for every c . Therefore the matrix $[\bar{v}_{ij}]$ is symmetric negative definite.

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