AN EVEN SIMPLER DIFFERENTIAL
DEMAND SYSTEM

E. Antony Selvanathan

University of Jaffna, Jaffna

and

Murdoch University, Western Australia

Discussion Paper 85.08

July 1985

ISSN 0811-6067

ISBN 0 86422 721 3

May 1985

AN EVEN SIMPLER DIFFERENTIAL DEMAND SYSTEM

by

E. Antony Selvanathan*

University of Jaffna, Jaffna

and

Murdoch University, Western Australia

Abstract: Keller (1984) proposed three simple variants of the well-known Rotterdam demand model. This paper proposes an even simpler differential demand system which has some attractive properties and is easy to estimate. The new model is estimated with Dutch and British data.

* I am indebted to Professor Kenneth W. Clements of the University of Western Australia for his guidance and comments and to Dr. Kenneth J. Harrison of Murdoch University for his help.

We write p_i , q_i for the price and quantity of good i ($i=1, \dots, n$), $M = \sum_{i=1}^n p_i q_i$ for income and $w_i = p_i q_i / M$ for the i^{th} budget share. Under general conditions the relative price version of the differential demand equation for good i can be written as (see Theil, 1980),

$$(1) \quad w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1}^n v_{ij} d(\log \frac{p_j}{P'}),$$

where $\theta_i = \partial(p_i q_i) / \partial M$ is the i^{th} marginal share; $d(\log Q) = \sum_{i=1}^n w_i d(\log q_i)$ is the Divisia volume index of the change in the consumer's real income; $d[\log (p_j / P')]$ is the change in the deflated price of j , where $d(\log P') = \sum_{i=1}^n \theta_i d(\log p_i)$ is the Frisch price index; and

$$(2) \quad v_{ij} = \frac{\lambda}{M} p_i u^{ij} p_j$$

is the $(i, j)^{\text{th}}$ price coefficient, where λ is the marginal utility of income and u^{ij} is the $(i, j)^{\text{th}}$ element of the inverse of the Hessian of the utility function $U = [\partial^2 u / \partial q_i \partial q_j]$. These v_{ij} 's satisfy

$$(3) \quad \sum_{j=1}^n v_{ij} = \phi \theta_i \quad i=1, \dots, n,$$

where $\phi = [\partial(\log \lambda) / \partial(\log M)]^{-1}$ is the income flexibility. It is to be noted that the coefficients of (1) need not be constants.

The absolute price version of (1) is

$$(4) \quad w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1}^n \pi_{ij} d(\log p_j),$$

where $\pi_{ij} = v_{ij} - \phi \theta_i \theta_j$ is the $(i,j)^{th}$ Slutsky coefficient. The n^2 Slutsky coefficients satisfy demand homogeneity

$$(5) \quad \sum_{j=1}^n \pi_{ij} = 0 \quad i=1, \dots, n$$

and Slutsky symmetry

$$(6) \quad \pi_{ij} = \pi_{ji} \quad i, j=1, \dots, n.$$

Dividing (4) by w_i , we obtain θ_i/w_i as the income elasticity of demand for i and π_{ij}/w_i as the $(i,j)^{th}$ price elasticity. We write these as

$$(7) \quad \eta_i = \frac{\theta_i}{w_i}, \quad \eta_{ij} = \frac{\pi_{ij}}{w_i}.$$

We propose the simple hypothesis that the larger the budget share of commodity j , the larger is the possibility of the consumer switching from commodity j to another commodity i in response to an increase in the j^{th} price; i.e., the larger is the budget share of j , the higher is the probability that i and j are substitutes. In other words, the price elasticity η_{ij} is proportional to the budget share w_j ,

$$(8) \quad \eta_{ij} = \alpha_i w_j \quad i \neq j,$$

where α_i is an i -subscripted positive constant. In view of (7), (8) implies $\pi_{ij} = \alpha_i w_i w_j$ for $i \neq j$. It follows from the previous equation and (5) that $-\pi_{ii} = \sum_{j \neq i} \pi_{ij} = \sum_{j \neq i} \alpha_i w_i w_j = \alpha_i w_i (1 - w_i)$; thus

$$(9) \quad \alpha_i = \frac{-\pi_{ii}}{w_i(1 - w_i)} = \frac{-\eta_{ii}}{(1 - w_i)}.$$

Equation (6) implies that $\alpha_i w_i w_j = \alpha_j w_j w_i$, from which it follows that

$$(10) \quad \alpha_i = \sigma,$$

a positive constant independent of i .

All the Slutsky coefficients can thus be expressed as $\pi_{ij} = \kappa w_i (\delta_{ij} - w_j)$, where $\kappa = -\sigma$ and δ_{ij} is the Kronecker delta, or in matrix form as

$$(11) \quad \pi = \kappa(W - ww'),$$

where $\pi = [\pi_{ij}]$, $W = \text{diag}[w_1, \dots, w_n]$ and $w = [w_i]$. As $\kappa < 0$, we see that $\pi_{ij} = -\kappa w_i w_j > 0$ for $i \neq j$, so that all the goods are substitutes in the Hicksian sense. It can be easily shown that π is negative semi-definite when $\sigma > 0$, which is in agreement with our hypothesis. As $\pi_{ij}/w_i w_j = \sigma$ for $i \neq j$, we can interpret σ as the cross-elasticity of substitution. Our hypothesis leads to the conclusion that this elasticity is independent of i and j ; this restriction has been previously used in applied work by Leser (1960) for consumer demand systems and by Powell and Gruen (1968) for output supply systems.

Using the definition of π_{ij} below equation (4), we can write $v_{ij} = \pi_{ij} + \phi \theta_i \theta_j$, or $v = \pi + \phi \theta \theta'$ in matrix form, where $v = [v_{ij}]$ and $\theta = [\theta_i]$. Using (11), we obtain

$$(12) \quad v = \kappa(W - ww') + \phi \theta \theta'.$$

The i^{th} diagonal element of v is $v_{ii} = \kappa w_i - \kappa w_i^2 + \phi \theta_i^2$; or equivalently, $v_{ii} = \phi w_i^2 (\tau [(1/w_i) - 1] + \eta_i^2)$, where $\tau = \kappa/\phi$. As $\phi < 0$, $\kappa < 0$ and $(1/w_i) - 1 > 0$, we have $v_{ii} < 0$. For $i \neq j$, $v_{ij} = \phi w_i w_j (\eta_i \eta_j - \tau)$, indicating that i and j are specific substitutes (complements) according to $\eta_i \eta_j < \tau$ ($> \tau$), where this definition of substitutability is from Houthakker (1960). From (2) we can also write $v = (\lambda/M) P^* U^{-1} P^*$, where $P^* = \text{diag}[p_1, \dots, p_n]$. Substituting this for v in (12) and taking the

inverse of both sides, we can show that the $(i,j)^{th}$ element of $P^{*-1}UP^{*-1}$ (the Hessian matrix of the utility function in expenditure terms) is

$$(13) \quad \frac{\partial^2 u}{\partial(p_i q_i) \partial(p_j q_j)} = \alpha + \beta \left(\frac{\delta_{ij}}{w_i} - (\eta_i + \eta_j) \right),$$

where $\alpha = (\lambda/M\kappa^2)a_{11} < 0$; $\beta = \lambda/M\kappa < 0$; $a_{11} = \kappa(\kappa + \phi \sum_i \theta_i \eta_i)/\phi < 0$. It can be easily verified that $\partial^2 u / \partial(p_i q_i) \partial(p_j q_j) > 0$ iff $\tau < \sum_k \theta_k^* \eta_k$, where $\theta_k^* = -\theta_k$ for $k \neq i, j$ and $\theta_k^* = (1 - \theta_k)$ for $k = i, j$, so that $\sum_k \theta_k^* = 1$. For the derivation of (13), see the Appendix.

For estimation we make the following adjustments to (4). First, we substitute $\kappa w_i (\delta_{ij} - w_j)$ for π_{ij} . Second, we replace w_i with its arithmetic average over the years $t-1$ and t , $\bar{w}_{it} = 1/2 (w_{it} + w_{i,t-1})$. Third, we replace the infinitesimal logarithmic change in a variable $d(\log x)$ with its finite log-change from $t-1$ to t , $Dx_t = \log x_t - \log x_{t-1}$. Fourth, we take κ to be a constant coefficient and use Working's (1943) model for the marginal shares so that $\theta_i = \beta_i + \bar{w}_{it}$, where β_i is a constant with $\sum_{i=1}^n \beta_i = 0$. The substitution term of (4) in terms of finite changes is thus

$$\sum_{j=1}^n \kappa \bar{w}_{it} (\delta_{ij} - \bar{w}_{jt}) Dp_{jt} = \kappa \bar{w}_{it} (Dp_{it} - DP_t) = \kappa \bar{w}_{it} Dp_{it}^*,$$

where $DP_t = \sum_{i=1}^n \bar{w}_{it} Dp_{it}$ is the Divisia price index and $Dp_{it}^* = Dp_{it} - DP_t$ is the change in the relative price of i . The estimating form of the i^{th} demand equation is thus

$$(14) \quad \bar{w}_{it} (Dq_{it} - DQ_t) = \beta_i DQ_t + \kappa \bar{w}_{it} Dp_{it}^* + \varepsilon_{it}, \quad i=1, \dots, n, \\ t=1, \dots, T,$$

where $DQ_t = \sum_{i=1}^n \bar{w}_{it} Dq_{it}$ is the finite-change Divisia volume index and

ε_{it} is a zero-mean disturbance. It is to be noted that only the own (Divisia) deflated price appears in each equation; and that (14) is linear in the parameters. Both are highly attractive properties of the model.

Summing both sides of (14) over $i=1, \dots, n$, we get $\sum_{i=1}^n \varepsilon_{it} = 0$, which indicates that one of the n equations can be deleted [see Barten (1969)]; we delete the n^{th} . We define the $(n-1)T$ -vector $y = [f'_1 f'_2 \dots f'_{n-1}]'$, with y_k as the k^{th} element, where $f_i = [\bar{w}_{i1}(Dq_{i1} - DQ_1) \dots \bar{w}_{iT}(Dq_{iT} - DQ_T)]'$ is a T -vector; the $(n-1)T$ -vector x_{1i} which has $DQ_1, \dots, DQ_T$ as elements $(T \times i) + 1$ to $T(1+i)$ and zeros elsewhere, with x_{1ik} the k^{th} element of x_{1i} ; and the $(n-1)T$ -vector $x_2 = [g'_1 g'_2 \dots g'_{n-1}]'$, with k^{th} element x_{2k} , where $g_i = [\bar{w}_{i1}^* Dp_{i1}^* \dots \bar{w}_{iT}^* Dp_{iT}^*]'$ is a T -vector. We can then write (14) as

$$(15) \quad y_k = \sum_{j=1}^{n-1} \beta_j x_{1jk} + \kappa x_{2k} + \varepsilon_k, \quad k=1, \dots, (n-1)T.$$

Thus as Keller (1984) proposed, (15) can be estimated by OLS by regressing y_k on the n explanatory variables $x_{11k}, \dots, x_{1,n-1,k}$ and x_{2k} . Although this is a simple procedure, the estimates are not efficient as the disturbances are non-spherical. We thus need to develop the GLS estimator.

We write (14) for $i=1, \dots, n-1$ as

$$(16) \quad y_t = X_t \delta + \varepsilon_t, \quad t=1, \dots, T,$$

where $y_t = [\bar{w}_{1t}(Dq_{1t} - DQ_t) \dots \bar{w}_{(n-1)t}(Dq_{(n-1)t} - DQ_t)]'$ is an $(n-1)$ -vector; X_t is a matrix of order $(n-1) \times n$ of which the i^{th} row is made up of only two non-zero elements, DQ_t in the i^{th} column and $\bar{w}_{it}^* Dp_{it}^*$ in the n^{th} column; $\delta = [\beta_1 \dots \beta_{n-1} \kappa]'$; and $\varepsilon_t = [\varepsilon_{1t} \dots \varepsilon_{(n-1)t}]'$.

A simple and plausible covariance structure is

$$(17) \quad \text{var } \varepsilon_t = \lambda^2 \Omega_t, \quad \Omega_t = W_t^* - W_t^* W_t^{*-1} W_t^{*'}, \quad \text{cov}(\varepsilon_t, \varepsilon_s) = 0 \text{ for } t \neq s,$$

where λ^2 is a constant, $W_t^* = \text{diag}[\bar{w}_{1t}, \bar{w}_{2t}, \dots, \bar{w}_{n-1,t}]$ and $W_t^{*'} = [\bar{w}_{1t} \ \bar{w}_{2t} \ \dots \ \bar{w}_{n-1,t}]'$. It is to be noted that under (17), $\text{var } \varepsilon_t$ is proportional to the $(n-1) \times (n-1)$ submatrix of the Slutsky matrix π . This result is implied by the theory of rational random behaviour [see Theil (1975/76, 1980)]. Under (17), the GLS estimator of δ

$$(18) \quad \hat{\delta} = \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} X_t \right\}^{-1} \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} y_t \right\}$$

and its variance is

$$(19) \quad \text{var } \hat{\delta} = \lambda^2 \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} X_t \right\}^{-1}.$$

The parameter λ^2 can be estimated consistently by

$$(20) \quad \hat{\lambda}^2 = \frac{1}{(n-1)T} \sum_{t=1}^T \hat{\varepsilon}_t' \Omega_t^{-1} \hat{\varepsilon}_t,$$

where $\hat{\varepsilon}_t = y_t - X_t \hat{\delta}$ is the $(n-1)$ -vector of GLS residuals.

Consider the special case when each budget share in Ω_t is replaced by its sample mean, so that this covariance matrix becomes a constant.

The scalar forms of (18)-(19) are then

$$(21) \quad \hat{\kappa} = (Z_0' Z_{QQ} - B) / A Z_{QQ}, \quad \hat{\beta}_i = (Z_{y_i Q} - \hat{\kappa} Z_{p_i Q}) / Z_{QQ}, \quad i=1, \dots, n-1,$$

and

$$(22) \quad \text{var } \hat{\kappa} = \hat{\lambda}^2 / A, \quad \text{var } \hat{\beta}_i = \hat{\lambda}^2 (A Z_{QQ} \bar{w}_i (1 - \bar{w}_i) + Z_{p_i Q}^2) / A Z_{QQ}^2, \quad i=1, \dots, n-1,$$

where $z_{QQ} = \sum_{t=1}^T DQ_t^2$, $z_{p_i Q} = \sum_{t=1}^T \bar{w}_{it} Dp_{it}^* DQ_t$, $z_0 = \sum_{t=1}^T \sum_{j=1}^n \bar{w}_{jt} Dp_{jt}^* y_{jt} / \bar{w}_j$,

$y_{it} = \bar{w}_{it} (Dq_{it} - DQ_t)$, $A = \sum_{i=1}^n \{ \sum_{t=1}^T (\bar{w}_{it} Dp_{it}^*)^2 - (z_{p_i Q}^2 / z_{QQ}) \} / \bar{w}_i$,

$z_{y_j Q} = \sum_{t=1}^T y_{jt} DQ_t$, $B = \sum_{j=1}^n z_{p_j Q} z_{y_j Q} / \bar{w}_j$ and $\bar{w}_i$ is the sample mean of $\bar{w}_{it}$. For the derivation of (21) and (22), see the Appendix.

As an illustrative application, we use the annual Dutch and British data from Theil (1975/76, Tables 5.2 and 5.3) to estimate the model by GLS under (17). The Dutch data cover the period 1922 to 1963 with war years excluded ($T=31$) and the British 1900 to 1938 with war years excluded ($T=30$). There are the same $n=4$ goods in both data sets. Following Theil (1975/76), for the U.K. we added a constant term to each equation to allow for trend-like changes in tastes etc. The estimates are (standard errors in parantheses):

	Netherlands	U.K.
Constant		
Food	-	.028 (.069)
Beverages	-	-.226 (.047)
Durables	-	-.001 (.057)
Remainder	-	.199 (.070)
β_i		
Food	-.112 (.027)	-.154 (.042)
Beverages	-.012 (.018)	.057 (.028)
Durables	.222 (.026)	.102 (.036)
Remainder	-.098 (.029)	-.004 (.043)
σ	.296 (.078)	.560 (.062)

The estimates of σ are close to the unweighted averages of the off-diagonal elasticities of substitution at means of .324 and .581, respectively, obtained from the symmetry-constrained estimates given in

Theil (1975/76, Tables 5.6 and 5.7). When we specify the covariance matrix as a constant, the estimation results remained almost unchanged; for full details, see the Appendix.

REFERENCES

- Barten, A.P. (1969). "Maximum Likelihood Estimation of a Complete System of Demand Equations," European Economic Review 1: 7-73.
- Houthakker, H.S. (1960). "Additive Preferences," Econometrica 28: 244-57.
- Keller, W.J. (1984). "Some Simple but Flexible Differential Consumer Demand Systems," Economics Letters 16: 77-82.
- Leser, C.E.V. (1960). "Demand Functions for Nine Commodity Groups in Australia," Australian Journal of Statistics 2: 102-13.
- Powell, A.A. and F.H.G. Gruen (1968). "The Constant Elasticity of Transformation Production Frontier and Linear Supply System," International Economic Review 9: 315-28.
- Theil, H. (1975/76). Theory and Measurement of Consumer Demand. Two Vols. Amsterdam: North-Holland.
- (1980). The System-Wide Approach to Microeconomics. Chicago: The University of Chicago Press.
- Working, H. (1943). "Statistical Laws of Family Expenditure," Journal of the American Statistical Association 38: 43-56.

APPENDIX

A Result on Partitioned Inversion**Proposition:**

Let D be a non-singular matrix of order m , and consider

$$X = D + \alpha uu' + \beta vv',$$

where α and β are scalars and u and v are two m -element vectors. Then

$$X^{-1} = D^{-1} + D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)',$$

where $a_{11} = -\alpha(1 + \beta\mu)/r$, $a_{21} = \alpha\beta\delta/r = a_{12}$, $a_{22} = -\beta(1 + \alpha\gamma)/r$,
 $r = (1 + \alpha\gamma)(1 + \beta\mu) - \alpha\beta\delta^2$, $\gamma = u'D^{-1}u$, $\delta = u'D^{-1}v$ and $\mu = v'D^{-1}v$,
 provided $r \neq 0$.

Proof: We shall verify that $XX^{-1} = I$. We have

$$\begin{aligned} XX^{-1} &= (D + \alpha uu' + \beta vv')[D^{-1} + D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)'] \\ &= I + u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + v(a_{21}D^{-1}u + a_{22}D^{-1}v)' + \alpha uu'D^{-1} \\ &\quad + \alpha uu'D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + \alpha uu'D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)' + \beta vv'D^{-1} \\ &\quad + \beta vv'D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + \beta vv'D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)' \\ &= I + (a_{11} + \alpha + \alpha\gamma a_{11} + \alpha\delta a_{21})uu'D^{-1} + (a_{12} + \alpha\gamma a_{12} + \alpha\delta a_{22})uv'D^{-1} \\ &\quad + (a_{21} + \beta\delta a_{11} + \beta\mu a_{21})vu'D^{-1} + (a_{22} + \beta + \beta\delta a_{12} + \beta\mu a_{22})vv'D^{-1} \\ &= I + k_1uu'D^{-1} + k_2uv'D^{-1} + k_3vu'D^{-1} + k_4vv'D^{-1}, \end{aligned}$$

where

$$k_1 = a_{11} + \alpha + \alpha\gamma a_{11} + \alpha\delta a_{21},$$

$$k_2 = a_{12} + \alpha\gamma a_{12} + \alpha\delta a_{22},$$

$$k_3 = a_{21} + \beta\delta a_{11} + \beta\mu a_{21}$$

and

$$k_4 = a_{22} + \beta + \beta\delta a_{12} + \beta\mu a_{22}.$$

Now we will verify that $k_1 = k_2 = k_3 = k_4 = 0$.

Consider

$$\begin{aligned} k_1 &= \alpha + a_{11}(1 + \alpha\gamma) + \alpha\delta a_{21} \\ &= \alpha - \alpha(1 + \beta\mu)(1 + \alpha\gamma)/r + \alpha\delta\alpha\beta\delta/r \\ &= \alpha - \alpha[(1 + \alpha\gamma)(1 + \beta\mu) - \alpha\beta\delta^2]/r \\ &= \alpha - \alpha \\ &= 0, \end{aligned}$$

$$\begin{aligned} k_2 &= a_{12} + \alpha\gamma a_{12} + \alpha\delta a_{22} \\ &= \alpha\beta\delta(1 + \alpha\gamma)/r - \alpha\delta\beta(1 + \alpha\gamma)/r \\ &= 0, \end{aligned}$$

$$\begin{aligned} k_3 &= a_{21}(1 + \beta\mu) + \beta\delta a_{11} \\ &= \alpha\beta\delta(1 + \beta\mu)/r - \beta\delta\alpha(1 + \beta\mu)/r \\ &= 0, \end{aligned}$$

$$\begin{aligned}
k_4 &= \beta + (1 + \beta\mu)a_{22} + \beta\delta a_{12} \\
&= \beta - \beta(1 + \beta\mu)(1 + \alpha\gamma)/r + \beta\delta\alpha\beta\delta/r \\
&= \beta - \beta[(1 + \alpha\gamma)(1 + \beta\mu) - \alpha\beta\delta^2]/r \\
&= \beta - \beta \\
&= 0.
\end{aligned}$$

Therefore $XX^{-1} = I$. This completes the proof.

Derivation of Equation (13)

From equation (12), we have

$$v = \kappa W - \kappa w w' + \phi \theta \theta'$$

where W is a non-singular diagonal matrix of order n and w, θ are n -vectors. Thus using the above proposition,

$$\begin{aligned}
v^{-1} &= \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} W^{-1} w (a_{11} W^{-1} w + a_{12} W^{-1} \theta)' + \frac{1}{\kappa^2} W^{-1} \theta (a_{21} W^{-1} w + a_{22} W^{-1} \theta)' \\
&= \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} \iota (a_{11} \iota' + a_{12} \eta)' + \frac{1}{\kappa^2} \eta (a_{21} \iota' + a_{22} \eta)', \\
&= \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} a_{11} \iota \iota' + \frac{1}{\kappa^2} a_{12} (\iota \eta' + \eta \iota') + \frac{1}{\kappa^2} a_{22} \eta \eta',
\end{aligned}$$

where $\iota = [1 \ 1 \ \dots \ 1]'$, $\eta = [\eta_1 \ \eta_2 \ \dots \ \eta_n]'$, $a_{11} = \kappa(1 + \frac{1}{\kappa} \phi \theta' W^{-1} \theta)/r$
 $= (\kappa + \phi \sum_{i=1}^n \theta_i \eta_i)/r$, $a_{12} = -\phi w' W^{-1} \theta/r = -\phi/r = a_{21}$, $a_{22} = -\phi(1 - w' W^{-1} w)/r$
 $= 0$ and $r = (1 - w' W^{-1} w)(1 + \phi \theta' W^{-1} \theta) + \kappa \phi (w' W^{-1} \theta/k)^2 = \phi/\kappa \neq 0$. Thus we have

$$v^{-1} = \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} a_{11} \iota \iota' - \frac{1}{\kappa} (\iota \eta' + \eta \iota').$$

Therefore if we denote $v^{-1} = [v^{ij}]$, we have

$$v^{ij} = \frac{1}{\kappa} a_{11} + \frac{1}{\kappa w_i} \delta_{ij} - \frac{1}{\kappa} (\eta_i + \eta_j).$$

From equation (2), we can write this as

$$\frac{M}{\lambda} \frac{1}{p_i p_j} \frac{\partial^2 u}{\partial q_i \partial q_j} = \frac{1}{\kappa} a_{11} + \frac{1}{\kappa w_i} \delta_{ij} - \frac{1}{\kappa} (\eta_i + \eta_j).$$

Therefore,

$$\frac{\partial^2 u}{\partial (p_i q_i) \partial (p_j q_j)} = \alpha + \frac{\beta}{w_i} \delta_{ij} - \beta (\eta_i + \eta_j),$$

where $\alpha = (\lambda/M\kappa^2) a_{11}$; $a_{11} = \kappa(\kappa + \phi \sum_i \theta_i \eta_i)/\phi$; and $\beta = \lambda/M\kappa$. The above is equation (13). Consider the case $i \neq j$,

$$\frac{\partial^2 u}{\partial (p_i q_i) \partial (p_j q_j)} = \alpha - \beta (\eta_i + \eta_j).$$

Therefore

$$\begin{aligned} \frac{\partial^2 u}{\partial (p_i q_i) \partial (p_j q_j)} > 0 & \quad \text{iff} \quad \alpha > \beta (\eta_i + \eta_j) \\ & \quad \text{iff} \quad \frac{\alpha}{\beta} < (\eta_i + \eta_j), \quad \text{as } \beta < 0 \\ & \quad \text{iff} \quad \kappa + \phi \sum_i \theta_i \eta_i > \phi (\eta_i + \eta_j) \quad \text{as } \phi < 0 \\ & \quad \text{iff} \quad \kappa > \phi \left[- \sum_i \theta_i \eta_i + (\eta_i + \eta_j) \right] \\ & \quad \text{iff} \quad \tau < \sum_i \theta_i^* \eta_i, \end{aligned}$$

where $\tau = \kappa/\phi$ and

$$\theta_k^* = \begin{cases} -\theta_k & \text{if } k \neq i, j \\ (1 - \theta_k) & \text{if } k = i, j, \end{cases}$$

so that $\sum_k \theta_k^* = 1$.

Derivation of (21)-(22)

From equations (18)-(19) we have

$$\hat{\delta} = \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} X_t \right\}^{-1} \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} y_t \right\}, \quad \text{var } \hat{\delta} = \lambda^2 \left\{ \sum_{t=1}^T X_t' \Omega_t^{-1} X_t \right\}^{-1}.$$

We replace each budget share in Ω_t with its sample mean,

$$(A1) \quad \Omega = \bar{W} - \bar{w}\bar{w}',$$

where $\bar{W} = \text{diag}[\bar{w}_1, \bar{w}_2, \dots, \bar{w}_{n-1}]$ and $\bar{w} = [\bar{w}_1 \ \bar{w}_2 \ \dots \ \bar{w}_{n-1}]$. If we take the inverse of both sides of (A1), we get

$$\Omega^{-1} = \bar{W}^{-1} + \frac{1}{\bar{w}_n} \bar{w}\bar{w}'.$$

Now we consider

$$\begin{aligned} X_t' \Omega^{-1} X_t &= \begin{pmatrix} DQ_t I_{n-1} \\ p_t^* \end{pmatrix} \Omega^{-1} [DQ_t I_{n-1} \quad p_t^*] \\ &= \begin{pmatrix} DQ_t \Omega^{-1} \\ p_t^{*'} \Omega^{-1} \end{pmatrix} [DQ_t I_{n-1} \quad p_t^*] \\ &= \begin{pmatrix} DQ_t^2 \Omega^{-1} & DQ_t \Omega^{-1} p_t^* \\ DQ_t p_t^{*'} \Omega^{-1} & p_t^{*'} \Omega^{-1} p_t^* \end{pmatrix}, \end{aligned}$$

where $p_t^* = [\bar{w}_{it} D p_{it}^*]$ is a $(n-1)$ -vector. Therefore

$$(A2) \quad \sum_{t=1}^T X_t' \Omega^{-1} X_t = \begin{bmatrix} Z_{QQ} \Omega^{-1} & \Omega^{-1} Z_{PQ} \\ Z_{PQ}' \Omega^{-1} & p^{**} \end{bmatrix},$$

where $Z_{QQ} = \sum_{t=1}^T DQ_t^2$; $Z_{PQ} = [Z_{P_1Q} \ Z_{P_2Q} \ \dots \ Z_{P_{n-1}Q}]'$, $Z_{P_iQ} = \sum_{t=1}^T \bar{w}_{it} D p_{it}^* DQ_t$

and $p^{**} = \sum_{t=1}^T p_t^{*'} \Omega^{-1} p_t^* = \sum_{i=1}^n \sum_{t=1}^T (\bar{w}_{it} D p_{it}^*)^2 / \bar{w}_i$.

Taking the inverse of both sides of (A2) gives

$$(A3) \quad \left\{ \sum_{t=1}^T X_t' \Omega^{-1} X_t \right\}^{-1} = \begin{bmatrix} \Omega/Z_{QQ} + Z_{PQ} Z_{PQ}' / A Z_{QQ}^2 & - Z_{PQ} / A Z_{QQ} \\ - Z_{PQ}' / A Z_{QQ} & 1/A \end{bmatrix},$$

where $A = p^{**} - (Z_{PQ}' \Omega^{-1} Z_{PQ}) / Z_{QQ} = \sum_{i=1}^n \{ \sum_{t=1}^T (\bar{w}_{it} D p_{it}^*)^2 - Z_{P_iQ}^2 / Z_{QQ} \} / \bar{w}_i$.

Similarly,

$$X_t' \Omega^{-1} y_t = \begin{bmatrix} DQ_t \Omega^{-1} y_t \\ p_t^{*'} \Omega^{-1} y_t \end{bmatrix}$$

which gives

$$(A4) \quad \sum_{t=1}^T X_t' \Omega^{-1} y_t = \begin{bmatrix} \Omega^{-1} Z_{yQ} \\ \sum_{t=1}^T p_t^{*'} \Omega^{-1} y_t \end{bmatrix},$$

where $Z_{yQ} = [Z_{y_1Q} \ \dots \ Z_{y_{n-1}Q}]'$, $Z_{y_iQ} = \sum_{t=1}^T y_{it} DQ_t$ and $y_{it} = \bar{w}_{it} (Dq_{it} - DQ_t)$.

Thus from (A3) and (A4) we can write

$$\hat{\delta} = \begin{bmatrix} \Omega/Z_{QQ} + Z_{PQ}Z'_{PQ}/AZ_{QQ}^2 & -Z_{PQ}/AZ_{QQ} \\ -Z'_{PQ}/AZ_{QQ} & 1/A \end{bmatrix} \begin{bmatrix} \Omega^{-1}Z_{yQ} \\ \sum_{t=1}^T P_t^* \Omega^{-1} y_t \end{bmatrix}.$$

As $\hat{\delta} = [\hat{\beta}' \ \hat{\kappa}']'$, we get

$$\begin{aligned} \hat{\kappa} &= -Z'_{PQ} \Omega^{-1} Z_{yQ} / AZ_{QQ} + \sum_{t=1}^T P_t^* \Omega^{-1} y_t / A \\ &= -B / AZ_{QQ} + Z_0 / A \\ (A5) \quad &= (Z_0 Z_{QQ} - B) / AZ_{QQ}, \end{aligned}$$

where $Z_0 = \sum_{t=1}^T P_t^* \Omega^{-1} y_t = \sum_{j=1}^n \{ \sum_{t=1}^T \bar{w}_{jt} D P_{jt}^* y_{jt} / \bar{w}_j \}$, $B = Z'_{PQ} \Omega^{-1} Z_{yQ} = \sum_{j=1}^n Z_{PjQ} Z_{y_jQ} / \bar{w}_j$; and

$$\begin{aligned} \hat{\beta} &= (\Omega/Z_{QQ} + Z_{PQ}Z'_{PQ}/AZ_{QQ}^2)(\Omega^{-1}Z_{yQ}) - Z_{PQ} \{ \sum_{t=1}^T P_t^* \Omega^{-1} y_t \} / AZ_{QQ} \\ &= Z_{yQ}/Z_{QQ} + Z_{PQ}Z'_{PQ} \Omega^{-1} Z_{yQ} / AZ_{QQ}^2 - Z_{PQ} \{ \sum_{t=1}^T P_t^* \Omega^{-1} y_t \} / AZ_{QQ} \\ &= Z_{yQ}/Z_{QQ} + BZ_{PQ}/AZ_{QQ}^2 - Z_0 Z_{PQ}/AZ_{QQ} \\ &= \{ Z_{yQ} - ((Z_0 Z_{QQ} - B)Z_{PQ}) / AZ_{QQ} \} / Z_{QQ} \\ &= (Z_{yQ} - \hat{\kappa} Z_{PQ}) / Z_{QQ}. \end{aligned}$$

Thus we have

$$(A6) \quad \hat{\beta}_i = (Z_{y_iQ} - \hat{\kappa} Z_{P_iQ}) / Z_{QQ} \quad \text{for } i=1, \dots, n-1.$$

Equations (A5) and (A6) are identical to equation (21) of the text.

Using (A3) we have

$$(A7) \quad \text{var } \hat{\delta} = \lambda^2 \begin{bmatrix} \Omega/Z_{QQ} + Z_{PQ}Z'_{PQ}/AZ_{QQ}^2 & -Z_{PQ}/AZ_{QQ} \\ -Z'_{PQ}/AZ_{QQ} & 1/A \end{bmatrix}.$$

Thus

$$(A8) \quad \text{var } \hat{\kappa} = \lambda^2/A$$

and

$$\text{var } \hat{\beta} = \lambda^2(\Omega/Z_{QQ} + Z_{PQ}Z'_{PQ}/AZ_{QQ}^2)$$

or equivalently,

$$(A9) \quad \text{var } \hat{\beta}_i = \lambda^2(AZ_{QQ}\bar{w}_i(1 - \bar{w}_i) + Z_{P_iQ}^2)/AZ_{QQ}^2, \quad i=1, \dots, n-1.$$

Equations (A8) and (A9) are equivalent to (22).

The Estimation Results

Below we present the estimates of the model with standard errors given in parantheses and where α_i is the constant term in equation i:

Under Specification (17)

Netherlands (without constants)

$$\beta_1 = -.112 (.027)$$

$$\beta_2 = -.012 (.018)$$

$$\beta_3 = .222 (.026)$$

$$\beta_4 = -.098 (.029)$$

$$\kappa = -.296 (.078)$$

U.K. (with constants)

$$\beta_1 = -.154 (.042) \quad \alpha_1 = .028 (.069)$$

$$\beta_2 = .057 (.028) \quad \alpha_2 = -.226 (.047)$$

$$\beta_3 = .102 (.036) \quad \alpha_3 = -.001 (.057)$$

$$\beta_4 = -.004 (.043) \quad \alpha_4 = .199 (.070)$$

$$\kappa = -.560 (.062)$$

Under Specification of Constant Covariance Matrix

$$\beta_1 = -.111 (.027)$$

$$\beta_2 = -.012 (.017)$$

$$\beta_3 = .222 (.026)$$

$$\beta_4 = -.099 (.029)$$

$$\kappa = -.291 (.078)$$

$$\beta_1 = -.151 (.042)$$

$$\beta_2 = .054 (.028)$$

$$\beta_3 = .104 (.036)$$

$$\beta_4 = -.007 (.043)$$

$$\kappa = -.568 (.061)$$

$$\alpha_1 = .032 (.068)$$

$$\alpha_2 = -.225 (.046)$$

$$\alpha_3 = -.009 (.058)$$

$$\alpha_4 = .201 (.070)$$