MORE ON AN AGGREGATIVE MULTIPRODUCT SUPPLY MODEL

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Abstract

Differential supply equations are expressed in terms of relative prices so that separability restrictions on technology can be easily incorporated. An illustrative application is provided.

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In a previous paper a simple revenue maximization framework was used to derive differential supply equations which resemble the absolute price version of the Rotterdam demand model [Clements (1980)]. Writing p_i and q_i for the price and quantity produced of product i ($i=1, \dots, n$), p and q for the corresponding vectors, y for real GDP, $Y = \sum_{i=1}^n p_i q_i$ for nominal GDP and $w_i = p_i q_i / Y$ for the share of i in GDP, the i^{th} supply equation can be expressed as

$$(1) \quad w_i d(\log q_i) = \theta_i d(\log y) + \sum_{j=1}^n \pi_{ij} d(\log p_j),$$

where θ_i is a normalized GDP coefficient proportional to $\partial(p_i q_i) / \partial y$ (see Appendix) and $\pi_{ij} = (p_i p_j / Y) (\partial q_i / \partial p_j)$ is a Slutsky coefficient. Using an obvious notation and with $W = \text{diag} [w_1, \dots, w_n]$, eq. (1) can be written in vector form as

$$(2) \quad W d(\log q) = \theta d(\log y) + \pi d(\log p).$$

The objective of this note is to extend the analysis to show how separability restrictions on the underlying technology can be incorporated in the model. This extension is closely allied to the relative price version of the Rotterdam model [Theil (1975/76, 1980)]. We also provide an illustrative application.

The revenue maximization problem is to choose q to maximize Y subject to the transformation function $f(q) = y$.¹ The first-order conditions consist of the transformation function and equality between price and marginal cost, $p = \lambda \partial f / \partial q$ where λ is a Lagrange multiplier. Differentiating these conditions with respect to p and y gives the matrix equation,

$$(3) \quad \begin{bmatrix} \lambda F & \partial f / \partial q \\ \partial f / \partial q' & 0 \end{bmatrix} \begin{bmatrix} \partial q / \partial p' & \partial q / \partial y \\ \partial \lambda / \partial p' & \partial \lambda / \partial y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0' & 1 \end{bmatrix},$$

where $F = \partial^2 f / \partial q \partial q'$ (symmetric positive definite); $\partial q / \partial p'$ is a matrix of price slopes of the supply functions; $\partial q / \partial y$ is a vector of slopes with respect to GDP; $\partial \lambda / \partial p'$ is a vector of price derivatives of the Lagrange multiplier; and

$\partial\lambda/\partial y$ is the GDP derivative. Eq. (3) is analogous to Barten's (1964) fundamental matrix equation for the consumer.²

Solving (3) for the matrix of price slopes gives (see Appendix)

$$(4) \quad \frac{\partial q}{\partial p'} = \lambda^{-1} F^{-1} + \frac{1}{\partial\lambda/\partial y} \frac{\partial q}{\partial y} \frac{\partial q'}{\partial y},$$

which shows that the substitution effect can be decomposed into specific and general components [Houthakker (1960)]. Using (4) the Slutsky matrix can be expressed as

$$(5) \quad \pi = P(\partial q/\partial p')P/Y = v - \phi\theta\theta',$$

where $P = \text{diag}[p]$; $v = P\lambda^{-1}F^{-1}P/Y$ is a price coefficient matrix which is inversely proportional to the Hessian of the transformation function in dollar terms;³ and $\phi = -\frac{\lambda^2/Y}{\partial\lambda/\partial y} > 0$ is a scalar parameter discussed in the Appendix. The row sums of v are proportional to the corresponding GDP coefficients, with ϕ the proportionality constant,

$$(6) \quad v1 = \phi\theta,$$

where $1 = [1, \dots, 1]'$ (see Appendix).

Defining the Frisch (marginal) price index $d(\log P^*) = \theta'd(\log p)$ and using (5), the substitution term in (2) becomes

$$\pi d(\log p) = v d(\log p) - \phi\theta d(\log P^*) = v d(\log p/P^*),$$

where the third step is based on (6). The i^{th} supply equation can then be written in terms of relative prices as

$$(7) \quad w_i d(\log q_i) = \theta_i d(\log y) + \sum_{j=1}^n v_{ij} d(\log p_j/P^*),$$

where $[v_{ij}] = v$.

Separability restrictions on $f(\cdot)$ translate directly into restrictions on v . Consider the special case in which the transformation function is additive in the n products, $f(q_1, \dots, q_n) = \sum_{i=1}^n f_i(q_i)$. Clearly, there is no interaction between products and the Hessian F and its inverse are both

diagonal. It follows that v is diagonal and from (6) that $v_{ii} = \phi\theta_i$.

Eq. (7) then becomes

$$(8) \quad w_i d(\log q_i) = \theta_i d(\log y) + \phi\theta_i d(\log p_i/P^*),$$

so that only the own-relative price appears. It is shown in the Appendix that in this case $\theta_i > 0$ (all i), implying that all products increase with GDP; i.e. all sectors behave in a procyclical manner. If we further assume that $f(\cdot)$ is homothetic (unitary GDP elasticities), then $\theta_i = w_i$ and (8) becomes

$$(9) \quad d(\log q_i/y) = \phi d(\log p_i/P^*)$$

and ϕ is interpreted as the own-price supply elasticity of each product and is the only unknown parameter.

We use (9) for our illustrative application to the output and price log-changes for the 112 industries of the ORANI model.⁴ There is one output and price change for each industry, so the data are cross-sectional. We estimate ϕ by regressing $\Delta \log q_i$ on a constant (representing $\Delta \log y$) and $\Delta \log p_i - \Delta \log P^*$. This yields $\hat{\phi} = .810$ with a standard error of .072. Thus the own-price supply elasticity is .81 which is estimated with a high degree of precision.

APPENDIX

In Clements (1980) the GDP coefficient for i is defined as

$\theta'_i = (Y/Y) p_i \partial q_i / \partial Y = (Y/Y) \partial (p_i q_i) / \partial Y$. Using the first-order condition to substitute $\lambda \partial f / \partial q_i$ for p_i and the constraint that $\sum_i (\partial f / \partial q_i) (\partial q_i / \partial Y) = 1$ (from the transformation function), we obtain $\sum_i \theta'_i = \lambda Y / Y$. The normalized GDP coefficient used in the text is $\theta_i = \theta'_i / \sum_j \theta'_j = (1/\lambda) \partial (p_i q_i) / \partial Y$, so that $\theta = [\theta_i] = (1/\lambda) P \partial q / \partial Y$. The θ_i 's clearly have a unit sum.

Writing $a = \partial f / \partial q$, the solution for the price and GDP derivatives in (3) is

$$\begin{bmatrix} \partial q / \partial p' & \partial q / \partial Y \\ \partial \lambda / \partial p' & \partial \lambda / \partial Y \end{bmatrix} = \frac{1}{a' \lambda^{-1} F^{-1} a} \begin{bmatrix} (a' \lambda^{-1} F^{-1} a) \lambda^{-1} F^{-1} - \lambda^{-1} F^{-1} a (\lambda^{-1} F^{-1} a)' & \lambda^{-1} F^{-1} \\ (\lambda^{-1} F^{-1} a)' & -1 \end{bmatrix}$$

so that

$$(A1) \quad \partial q / \partial p' = \lambda^{-1} F^{-1} - (a' \lambda^{-1} F^{-1} a)^{-1} \lambda^{-1} F^{-1} a (\lambda^{-1} F^{-1} a)'$$

$$(A2) \quad \partial q / \partial Y = (a' \lambda^{-1} F^{-1} a)^{-1} \lambda^{-1} F^{-1} a$$

$$(A3) \quad \partial \lambda / \partial p' = (a' \lambda^{-1} F^{-1} a)^{-1} (\lambda^{-1} F^{-1} a)'$$

$$(A4) \quad \partial \lambda / \partial Y = -(a' \lambda^{-1} F^{-1} a)^{-1}$$

Using (A4) in (A2) gives $\partial q / \partial Y = -(\partial \lambda / \partial Y) \lambda^{-1} F^{-1} a$ so that

$$(A5) \quad \lambda^{-1} F^{-1} a = -\frac{1}{\partial \lambda / \partial Y} \frac{\partial q}{\partial Y}$$

Using (A4) in (A1) gives

$$\partial q / \partial p' = \lambda^{-1} F^{-1} + (\partial \lambda / \partial Y) \lambda^{-1} F^{-1} a (\lambda^{-1} F^{-1} a)'$$

and substituting (A5) in this equation gives eq. (4).

Using the definitions of v [see below (5)] and θ (see above), pre-multiplying (4) by P and postmultiplying by P/Y gives

$$P(\partial q / \partial p') P/Y = v + \frac{1}{\partial \lambda / \partial Y} P \frac{\partial q}{\partial Y} \frac{\partial q'}{\partial Y} P/Y = v - \phi \theta \theta',$$

where $\phi = -\frac{\lambda^2/Y}{\partial \lambda / \partial Y}$. The above is eq. (5) of the text. To interpret ϕ , we write the first-order conditions as $p' = \lambda \partial f / \partial q'$ and postmultiply by q to give $Y = p'q = \lambda (\partial f / \partial q') q$. When $f(\cdot)$ is linear homogenous $(\partial f / \partial q') q = Y$,

so that $Y = \lambda y$, implying that the Lagrange multiplier is the GDP deflator. It follows that $\lambda^2/Y = \lambda/y$ so that $\phi = -(\partial \log \lambda / \partial \log y)^{-1}$ is minus the reciprocal of the GDP elasticity of λ . When the transformation function is not linear homogenous, ϕ does not have this interpretation; however, under additivity and homotheticity ϕ is still interpreted as the own-price supply elasticity of each product.

To derive eq. (6) we use the definition of v and the fact that $p = P_1$ to give $v_1 = P \lambda^{-1} F^{-1} p / Y = P \lambda^{-1} F^{-1} \lambda (\partial f / \partial q) / Y$, where the second step uses the first-order conditions. Using (A5) we obtain

$$v_1 = - \frac{\lambda/Y}{\partial \lambda / \partial y} P \frac{\partial q}{\partial y} = - \frac{\lambda^2/Y}{\partial \lambda / \partial y} \theta = \phi \theta ,$$

which is eq. (6).

It follows from (6) that, since $v' \theta = 1$, $\phi = v' v_1 > 0$ as v is positive definite. Under additivity (6) implies $v_{ii} = \phi \theta_i > 0$ and as ϕ is positive so is θ_i .

FOOTNOTES

1. The analysis can be also applied to the demand for factor inputs by treating them as outputs with negative prices. In this case revenue maximization is equivalent to direct profit maximization without a preliminary cost minimizing step [c.f. Laitinen and Theil (1978)].
2. For alternative analyses of the matrix equation in production theory, see Laitinen and Theil (1978) and Theil (1977).
3. C.f. Theil (1975/76, p. 29).
4. ORANI is a sophisticated large-scale general equilibrium model of the Australian economy [Dixon (1979)]. The data are from Dixon et. al. (1979); we use base-period value-added shares for the w_i 's.

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