HOLDINGS OF FINANCIAL ASSETS:
A MARKOV CHAIN ANALYSIS

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Discussion Paper 81.05
July 1981

Revised; July 1981

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Abstract

This paper introduces the Markov chain model as a simple tool for analysing the pattern of financial asset holdings over time. The model is based on transition probabilities which give the probability of switching \$1 of wealth from one asset to another. These probabilities can be estimated with aggregate time series data by conventional regression techniques. The estimates can be used to compute (i) the average time that wealth spends in each asset and (ii) the proportions of wealth held in the various assets in the steady state. The Markov model is shown to be closely related to the well-known multivariate stock adjustment model. Finally, an illustrative application with Australian data is provided.

* We would like to acknowledge the helpful comments of John Hawkins, Ulrich Kohli, Bill Norton, Glenn Stevens and Henri Theil. The views expressed herein are not necessarily those of the Reserve Bank of Australia.

1. Introduction

The pattern of financial asset holdings has a number of important policy implications. These include the macroeconomic phenomena such as the controllability and definition of the money supply, interaction of the real and monetary economy, disintermediation of the banking system, and so on. Accordingly, much research has been devoted to modelling financial asset holdings in terms of income, wealth, interest rates, etc.¹

In this paper we analyse asset holdings with an alternative approach, the Markov chain model. This approach, which has not been previously applied to asset holdings, has a number of attractive features, the main one being its simplicity. The model is implemented with conventional regression techniques and the parameter estimates give the probability of switching \$1 of financial wealth currently held in the form of one asset to another (e.g., from a deposit at a savings bank to a building society). This switching probability is obviously closely related to the degree of substitutability between the assets involved. Another interesting result of the Markov analysis is the estimated steady-state asset distribution, which gives the proportions of wealth held in the various assets in long-run equilibrium. This distribution can be used to answer questions such as the ultimate relative size of the non-bank financial intermediaries.

2. The Markov Chain Model²

Consider \$1 of financial wealth which can be held in any of n assets. The Markov chain model is a stochastic process describing where this \$1 will be held, with the probability of it being held in a certain asset depending only on where it was last held. Thus the Markov model specifies that where the wealth was held before the previous period is irrelevant for the current placement of the \$1.

The probability of switching wealth from one asset to another is constant in the Markov model. In a more general framework this probability could be a function of economic variables such as relative rates of return. Therefore, the assumption of constant probabilities is at least potentially a

strong restriction, one which would not be acceptable in all circumstances. However, we argue in Section 5 that this assumption is likely to be acceptable with our data.

Let $p_{ij} \geq 0$ be the probability that the \$1 will be held in the form of asset j given that it was previously in asset i ; we shall refer to p_{ij} as the $(i,j)^{\text{th}}$ transition probability. We assume that the \$1 will continue to be held in the form of one of the assets, implying $\sum_{j=1}^n p_{ij} = 1$ ($i=1, \dots, n$). Writing q_{i0} for the initial proportion of wealth held in i , $q_{i0}p_{ij}$ of this goes to asset j next period. The total proportion in j next period is the sum of this over i , $q_{j1} = \sum_i q_{i0}p_{ij}$, which we write in vector form as

$$(1) \quad q'_1 = q'_0 P,$$

where $q'_t = [q_{it}]$ is the t^{th} asset distribution ($t=0,1$) and $P=[p_{ij}]$ is the transition matrix.

For the subsequent period we have $q'_2 = q'_1 P = q'_0 P^2$ [from (1)]. Repeated application of this procedure yields $q'_t = q'_0 P^t$. The $(i,j)^{\text{th}}$ element of P^t thus is the probability of moving \$1 from asset i to j in t steps (periods). The i^{th} row of P^t is the t^{th} asset distribution if initially all wealth is held in asset i (as then q_0 has 1 as the i^{th} element and zeros elsewhere).

If P^t for some t consists exclusively of nonzero elements (implying that any transition can be made in at most t steps), then the Markov chain is said to be regular. In this case, it can be shown that no matter what initial distribution q_0 , there is always convergence to the same steady-state distribution $\pi = [\pi_i]$ as $t \rightarrow \infty$,

$$(2) \quad \lim_{t \rightarrow \infty} q'_0 P^t = \pi'.^3$$

Furthermore, the steady-state proportions π_i sum to one and are all positive. The vector π is the steady-state distribution in the sense that once it is attained, the system will stay there: $\pi' = \pi' P$. This does not mean that there is no movement of wealth between assets in the steady state, but that movements cancel each other out to keep the asset distribution constant.

For eq. (2) to hold, P^t converges as $t \rightarrow \infty$ to a matrix whose rows are all identical with the steady-state distribution,

inequality constrained maximum likelihood (ML) estimator derived by Lee et.al.(1970), which is based on the assumption that the errors have a multinomial distribution. This estimator takes account of the heteroscedasticity and the non-negativity restriction by using quadratic programming techniques.

4. Relationship between the Markov and Multivariate Stock Adjustment Models

The well-known multivariate stock adjustment model expresses the change in holdings of asset j in terms of the gap between desired and actual holdings of all the n assets in the portfolio (Brainard and Tobin, 1968). We can write this model in share form as $q_{jt} - q_{j,t-1} = \sum_{i=1}^n \alpha_{ij} (q_{it}^* - q_{i,t-1}) + \varepsilon_{jt}$, $j = 1, \dots, n$, where the α_{ij} 's are adjustment parameters, q_{it}^* is the desired holdings of asset i at time t and ε_{jt} is an error term. Substituting $q_{j,t-1} = 1 - \sum_{i \neq j} q_{i,t-1}$ and rearranging, this can be expressed as eq. (4) with $\beta_{0j} = \sum_{i=1}^n \alpha_{ij} q_{it}^* - \alpha_{jj} + 1$ and $\beta_{ij} = \alpha_{jj} - \alpha_{ij} - 1$ ($i \neq j$).

In the stock adjustment model the α_{ij} 's are typically treated as constant adjustment parameters, while the desired holdings q_{it}^* is postulated to be a function of economic variables such as relative rates of return. The constant α_{ij} 's imply that the β_{ij} 's ($i \neq j$) defined in the previous paragraph are also constant. To the extent that the economic variables determining q_{it}^* are more or less constant, β_{0j} of the previous paragraph can then also be treated as a constant. As the Markov and stock adjustment models can be both expressed in the form of eq. (4) with constant coefficients, it follows that the two models are identical in this situation and we can interpret $q_{it}^* = q_i^*$ as the steady-state proportion π_i of Sec. 2. The clear advantage of the Markov model is the interpretation of the parameters as transition probabilities, about which one can have some intuitive ideas.

5. Illustrative Application

Our data are quarterly Australian for the period September 1973 - June 1979 (24 observations). We distinguish $n = 5$ financial assets which are listed in Table 1.⁷ Over this period the relative rates of return for these assets have been more or less constant (Clements and Taylor, 1980), so that the assumption of constant transition probabilities is not too implausible.

$$\lim_{t \rightarrow \infty} P^t = \mathbf{1}\pi',$$

where $\mathbf{1}$ is an n -vector of unit elements. It follows that π can be computed by powering the transition matrix.

With q_j the proportion of wealth in asset j , next period $q_j p_{jj}$ of this stays in j , $q_j p_{jj}^2$ the period after, and so on. Accordingly, the total time that wealth stays in asset j is $T_j = q_j + q_j p_{jj} + q_j p_{jj}^2 + \dots$, and the average (or expected) time is

$$(3) \quad T_j / q_j = 1 / (1 - p_{jj}).^4$$

3. Estimation of the Transition Probabilities

In most applications of Markov chains data are available on the micro transitions from one state to another. In this case, the transition probabilities p_{ij} can be estimated very simply by the corresponding sample proportions, i.e. by the proportion of cases originating in state i which go to j in one step.⁵ However, in our application the only available data are the observed asset distributions through time q_t ; there is no data on the micro transitions. As a result, we need to use regression techniques to estimate the p_{ij} 's.⁶

From eq. (1) the proportion of wealth currently held in asset j is a linear function of the n proportions in the previous period,

$$q_{jt} = \sum_{i=1}^n p_{ij} q_{i,t-1}. \text{ As the proportions have a unit sum, we substitute } 1 - \sum_{i \neq j} q_{i,t-1} \text{ for } q_{j,t-1}; \text{ after adding an error term } \epsilon_{jt} \text{ we obtain}$$

$$(4) \quad q_{jt} = \beta_{0j} + \sum_{i \neq j} \beta_{ij} q_{i,t-1} + \epsilon_{jt},$$

where $\beta_{0j} = p_{jj}$ and $\beta_{ij} = p_{ij} - p_{jj}$. Eq. (4) is in the form of a linear regression equation and least squares can be used to estimate the p_{ij} 's. This estimator automatically satisfies the adding-up restriction $\sum_{j=1}^n p_{ij} = 1$, but not necessarily the non-negativity restriction $p_{ij} \geq 0$.

As the dependent variable of (4) is a proportion, the error ϵ_j is heteroscedastic. The least squares estimator is inefficient as it does not take account of this heteroscedasticity. In our application we employ the

TABLE 1

MAXIMUM LIKELIHOOD ESTIMATES OF TRANSITION PROBABILITIES:
 AUSTRALIA, SEPTEMBER 1973 - JUNE 1979
 (Standard errors in parentheses)

Previous asset (i of p_{ij})	Current asset (j of p_{ij})				
	1	2	3	4	5
1. Fixed deposits with trading banks	.733 (.253)	0	.166 (.361)	.102 (.343)	0
2. Savings bank investment accounts	.022 (.122)	.913 (.029)	.004 (.185)	.003 (.165)	.058 (.042)
3. Savings bank ordinary accounts	.020 (.037)	.053 (.012)	.908 (.048)	.019 (.045)	0
5. 4. Deposits with permanent building societies	.113 (.108)	0	0	.837 (.140)	.050 (.046)
5. 5. Special/Australian savings bonds	0	0	.073 (.177)	.200 (.193)	.727 (.105)

The ML estimates of the transition probabilities are given in Table 1.⁸ Each row of the table gives the probability of moving in one quarter \$1 previously held in the asset at the left of the row (asset i) into each of the five assets (asset j). According to the first row, for \$1 previously in a fixed deposit the probability of continuing to hold it in that form is .73. The probability of moving it into a savings bank investment account is zero; to a savings bank ordinary account .17; to a building society .10; and there is a zero probability of moving it into bonds. As the wealth must go somewhere, these probabilities sum to one. Each column of the table gives the probability of moving in one quarter \$1 into the asset given at the top of the column (asset j) from each of the five assets (asset i). According to the first column, the probabilities of moving \$1 into a fixed deposit are .73, .02, .02, .11 and 0, respectively.

It should be noted that these probabilities all refer to transitions over one quarter. Thus the zeros are to be interpreted as indicating that wealth is not transferred between the relevant assets over a one quarter period; they do not indicate that wealth is never transferred. E.g., although there is a zero probability of a transition from a fixed deposit to a savings bank investment account in one quarter ($p_{12}=0$), over two quarters such a transition can be made via a savings bank ordinary account. In the first quarter there is a .17 probability of a transition from a fixed deposit to an ordinary account ($p_{13}=.17$); in the next quarter there is a .05 probability of a transition from the ordinary to the investment account ($p_{32}=.05$). Accordingly, over the two quarters there is a $.17 \times .05 = .009$ probability of shifting \$1 of wealth from a fixed deposit to an investment account. More generally, it can be easily shown that any transition can be made in at most three quarters, so that the Markov chain is regular.

For a given row, the diagonal probability is always much larger than any of the other elements. This means that for each asset, the most likely one quarter "move" is no move; i.e. wealth is most likely to stay where it was previously. Thus, there is a good deal of inertia (or autocorrelation) in holdings of these assets. There are nonetheless a number of substantial off-diagonal

elements which reflect the (one quarter) interrelationships between assets. There are large cross-probabilities for fixed deposits to savings bank ordinary accounts (p_{13}), fixed deposits to building societies (p_{14}), building societies to fixed deposits (p_{41}), and bonds to building societies (p_{54}). It should be noted that although all diagonal probabilities are estimated quite precisely, the standard errors for most of the off-diagonal elements are relatively large. This is probably due to the lack of independent variation in the asset proportions over this period.⁹

The transition matrix has the interesting property that it is nearly symmetric. Of the ten pairs of off-diagonal elements p_{ij} , p_{ji} for $i \neq j$, only two exhibit substantial departure from pairwise equality, (i) fixed deposits, savings bank ordinary accounts ($p_{13} - p_{31} = .146$) and (ii) building societies, bonds ($p_{45} - p_{54} = -.150$). With the exception of these two pairs, we conclude that the probability of moving wealth from one asset to another in one quarter is approximately equal to that of doing the reverse transaction. This seems to be an attractive property of the matrix, one which has much intuitive plausibility.

Applying eq.(3), the average time wealth spends in each asset is (3.7, 11.5, 10.9, 6.1, 3.7) quarters (in the same order as Table 1). As can be seen, the turnover of the two types of savings bank deposits is half (or less) that of the other three assets. This corresponds with the notion that very little "smart money" is held in these low interest bearing forms; people who hold money in savings banks are not concerned with actively managing their wealth.

The steady-state asset distribution corresponding to the transition matrix is $\pi' = (.141, .203, .332, .235, .087)$. Thus in the steady-state, fixed deposits make up 14 per cent of financial wealth, savings bank investment accounts 20 per cent, savings bank ordinary accounts 33 per cent, building society deposits 24 per cent, with the remaining 9 per cent in bonds. Over the sample period the

mean asset proportions are (.127, .187, .410, .204, .072), so that the main differences of the steady-state distribution from the means are (i) the 8 percentage point fall in the savings bank ordinary account share and (ii) the rise in the building society share by 3 percentage points. It is of interest that the system is very close to the steady state by the end of the sample period (June 1979) when the observed asset distribution is (.140, .209, .334, .231, .087). This is to be expected if the data are generated by a process which can be well-approximated by a Markov chain; i.e. as we move through time the observed asset distribution should get closer to the steady state. Hence, this finding gives support to the Markov model.

In interpreting this steady-state distribution, it should be noted that it is intimately related to the transition probabilities, and as indicated above these probabilities are likely to depend on economic variables such as relative rates of return. Accordingly, the above steady-state distribution depends on the existing structure of returns. Although over the sample period relative rates of return did not vary a great deal, if in the future returns were to change, then so would the steady-state distribution (c.f. Sec. 4).

6. Concluding Comments

In this paper we have introduced the Markov chain model as a simple tool for analysing the pattern of financial asset holdings over time. This model is based on transition probabilities which give the probability of switching \$1 of wealth from one asset to another; these probabilities have a simple intuitive interpretation. The model can be estimated with aggregate time series data by conventional regression techniques. The estimated parameters can then be used to compute (i) the average time that wealth stays in each asset and (ii) the proportion of wealth held in the various assets in the steady state. We showed that the Markov model is closely related to the well-known multivariate stock adjustment model.

As an illustrative application, we estimated the model with quarterly data for five assets and the results were quite positive. We found that there was a good deal of inertia in holdings of these assets, but still some notice-

able interrelationships between assets. An interesting finding was that the probability of moving wealth from one asset to another (in one quarter) is approximately equal to that of doing the reverse transaction. This means that the pattern of asset holdings is nearly symmetric. Finally, we showed that the system was very close to the steady state by the end of the sample period, which provides an additional check on the appropriateness of the Markov model.

FOOTNOTES

1. For a brief survey and references, see Taylor and Clements (1980).
2. This section draws on Theil (1972, Chap.5). For a listing of applications of the Markov chain model in the social sciences, see Lee et.al. (1970).
3. See, e.g., Theil (1972, Sec.5.1).
4. See Prais (1955).
5. Anderson and Goodman (1957) show that this is the maximum likelihood estimator.
6. Lee et.al.(1970) provide an extensive survey of the various estimators which can be used for this type of problem.
7. The data are from Taylor and Clements (1980) and are listed in an appendix, available on request.
8. The multinomial error specification requires that the sample size (the total number of micro units) be specified for each observation. For this we use real per capita financial wealth, defined as the sum of holdings of the five assets with the CPI as deflator. These data are also listed in the appendix. For estimation we use the computer routine given in Lee et.al.(1970).
9. The standard errors given in Table 1 are only approximate and should be treated with due care. In an appendix (available on request) we set out in full detail the derivation of the (approximate) covariance matrix of the estimates.

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