AN EVEN SIMPLER DIFFERENTIAL
DEMAND SYSTEM

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Abstract: Keller (1984) proposed three simple variants of the well-known Rotterdam demand model. This paper proposes an even simpler differential demand system which has some attractive properties and is easy to estimate. The new model is estimated with Dutch and British data.

*I am indebted to Professor Kenneth W. Clements of the University of Western Australia for his guidance and comments and to Dr. Kenneth J. Harrison of Murdoch University for his help.
We write $p_i$, $q_i$ for the price and quantity of good $i$ ($i=1, \ldots, n$), $M = \sum_{i=1}^{n} p_i q_i$ for income and $w_i = p_i q_i / M$ for the $i$th budget share. Under general conditions the relative price version of the differential demand equation for good $i$ can be written as (see Theil, 1980),

\begin{equation}
(1) \quad w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1}^{n} v_{ij} d(\log \frac{p_j}{P'}),
\end{equation}

where $\theta_i = \frac{\partial (p_i q_i)}{\partial M}$ is the $i$th marginal share; $d(\log Q) = \sum_{i=1}^{n} w_i d(\log q_i)$ is the Divisia volume index of the change in the consumer's real income; $d[\log (p_j / P')]$ is the change in the deflated price of $j$, where $d(\log P') = \sum_{i=1}^{n} \theta_i d(\log p_i)$ is the Frisch price index; and

\begin{equation}
(2) \quad v_{ij} = \frac{\lambda}{M} p_i u_{ij} p_j
\end{equation}

is the $(i,j)^{th}$ price coefficient, where $\lambda$ is the marginal utility of income and $u_{ij}$ is the $(i,j)^{th}$ element of the inverse of the Hessian of the utility function $U = [\partial^2 u / \partial q_i \partial q_j]$. These $v_{ij}$'s satisfy

\begin{equation}
(3) \quad \sum_{j=1}^{n} v_{ij} = \phi \theta_i \quad i=1, \ldots, n,
\end{equation}

where $\phi = [\partial (\log \lambda) / \partial (\log M)]^{-1}$ is the income flexibility. It is to be noted that the coefficients of (1) need not be constants.

The absolute price version of (1) is

\begin{equation}
(4) \quad w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1}^{n} \pi_{ij} d(\log p_j),
\end{equation}
where $\pi_{ij} = v_{ij} - \phi_i \phi_j$ is the $(i,j)^{th}$ Slutsky coefficient. The $n^2$ Slutsky coefficients satisfy demand homogeneity

$$\sum_{j=1}^{n} \pi_{ij} = 0 \quad \text{for } i=1, \ldots, n$$

and Slutsky symmetry

$$\pi_{ij} = \pi_{ji} \quad \text{for } i,j=1, \ldots, n.$$

Dividing (4) by $w_i$, we obtain $\theta_i/w_i$ as the income elasticity of demand for $i$ and $\pi_{ij}/w_i$ as the $(i,j)^{th}$ price elasticity. We write these as

$$\eta_i = \frac{\theta_i}{w_i}, \quad \eta_{ij} = \frac{\pi_{ij}}{w_i}.$$

We propose the simple hypothesis that the larger the budget share of commodity $j$, the larger is the possibility of the consumer switching from commodity $j$ to another commodity $i$ in response to an increase in the $j^{th}$ price; i.e., the larger is the budget share of $j$, the higher is the probability that $i$ and $j$ are substitutes. In other words, the price elasticity $\eta_{ij}$ is proportional to the budget share $w_j$,

$$\eta_{ij} = \alpha_i w_j \quad \text{for } i \neq j,$$

where $\alpha_i$ is an $i$-subscripted positive constant. In view of (7), (8) implies $\pi_{ij} = \alpha_i w_j w_j$ for $i \neq j$. It follows from the previous equation and (5) that 

$$-\pi_{ii} = \sum_{j \neq i} \pi_{ij} = \sum_{j \neq i} \alpha_i w_i w_j = \alpha_i w_i (1 - w_i);$$

thus

$$\alpha_i = \frac{-\pi_{ii}}{w_i (1 - w_i)} = -\frac{\eta_{ii}}{(1 - w_i)}.$$

Equation (6) implies that $\alpha_i w_i w_j = \alpha_j w_j w_i$, from which it follows that

$$\alpha_i = \sigma_i.$$
a positive constant independent of i.

All the Slutsky coefficients can thus be expressed as

\[ \pi_{ij} = \kappa w_i (\delta_{ij} - w_j), \]

where \( \kappa = -\sigma \) and \( \delta_{ij} \) is the Kronecker delta, or in matrix form as

\[ (11) \quad \pi = \kappa (W - ww'), \]

where \( \pi = [\pi_{ij}] \), \( W = \text{diag}[w_1, \ldots, w_n] \) and \( w = [w_i] \). As \( \kappa < 0 \), we see that \( \pi_{ij} = -\kappa w_i w_j > 0 \) for \( i \neq j \), so that all the goods are substitutes in the Hicksian sense. It can be easily shown that \( \pi \) is negative semi-definite when \( \sigma > 0 \), which is in agreement with our hypothesis. As \( \pi_{ij}/w_i w_j = -\sigma \) for \( i \neq j \), we can interpret \( \sigma \) as the cross-elasticity of substitution. Our hypothesis leads to the conclusion that this elasticity is independent of \( i \) and \( j \); this restriction has been previously used in applied work by Leser (1960) for consumer demand systems and by Powell and Gruen (1968) for output supply systems.

Using the definition of \( \pi_{ij} \) below equation (4), we can write

\[ \nu_{ij} = \pi_{ij} + \phi \theta_i \theta_j, \]

or \( \nu = \pi + \phi \theta \theta' \) in matrix form, where \( \nu = [\nu_{ij}] \) and \( \theta = [\theta_i] \). Using (11), we obtain

\[ (12) \quad \nu = \kappa (W - ww') + \phi \theta \theta'. \]

The \( i \)th diagonal element of \( \nu \) is \( \nu_{ii} = \kappa w_i - \kappa w_i^2 + \phi \theta_i^2 \); or equivalently, \( \nu_{ii} = \phi w_i^2 (\tau[(1/w_i) - 1] + \eta_i^2) \), where \( \tau = \kappa/\phi \). As \( \phi < 0, \kappa < 0 \) and \( (1/w_i) - 1 > 0 \), we have \( \nu_{ii} < 0 \). For \( i \neq j \), \( \nu_{ij} = \phi w_i w_j (\eta_i \eta_j - \tau) \), indicating that \( i \) and \( j \) are specific substitutes (complements) according to \( \eta_i \eta_j < \tau \) (\( > \tau \)), where this definition of substitutability is from Houthakker (1960). From (2) we can also write \( \nu = (\lambda/M) P^* U^{-1} P^* \), where \( P^* = \text{diag}[p_1, \ldots, p_n] \). Substituting this for \( \nu \) in (12) and taking the
inverse of both sides, we can show that the $(i,j)^{th}$ element of $P^{*-1}UP^{*-1}$ (the Hessian matrix of the utility function in expenditure terms) is

$$
\frac{\partial^2 u}{\partial (p_i q_i) \partial (p_j q_j)} = \alpha + \beta \left( \frac{\delta i}{\delta i} - (\eta_i + \eta_j) \right),
$$

where $\alpha = (\lambda/MK^2)a_{11} < 0$; $\beta = \lambda/MK < 0$; $a_{11} = \kappa(\kappa + \phi \sum_i \theta_i \eta_i)/\phi < 0$. It can be easily verified that $\partial^2 u/\partial (p_i q_i) \partial (p_j q_j) > 0$ iff $\sum_i \theta_i^* \eta_i$, where $\theta_i^* = -\theta_i$ for $k \neq i,j$ and $\theta_i^* = (1 - \theta_i)$ for $k = i,j$, so that $\sum_i \theta_i^* = 1$.

For the derivation of (13), see the Appendix.

For estimation we make the following adjustments to (4). First, we substitute $\kappa w_i (\delta i - \delta j)$ for $\pi_{ij}$. Second, we replace $w_i$ with its arithmetic average over the years $t-1$ and $t$, $\overline{w}_it = \frac{1}{2} (w_{it} + w_{i,t-1})$. Third, we replace the infinitesimal logarithmic change in a variable $d(\log x)$ with its finite log-change from $t-1$ to $t$, $Dx_t = \log x_t - \log x_{t-1}$. Fourth, we take $\kappa$ to be a constant coefficient and use Working's (1943) model for the marginal shares so that $\theta_i = \beta_i + \bar{w}_{it}$, where $\beta_i$ is a constant with $\sum_i \beta_i = 0$. The substitution term of (4) in terms of finite changes is thus

$$
\sum_{j=1}^n \kappa \bar{w}_{it} (\delta i - \delta j)DP_{jt} = \kappa \bar{w}_{it} (DP_{it} - DP_t) = \kappa \bar{w}_{it} DP_{it},
$$

where $DP_t = \sum_{i=1}^n \bar{w}_{it} DP_{it}$ is the Divisia price index and $DP_{it} = DP_{it} - DP_t$ is the change in the relative price of $i$. The estimating form of the $i^{th}$ demand equation is thus

$$
\bar{w}_{it} (DQ_{it} - DQ_t) = \beta_i Q_i + \kappa \bar{w}_{it} DP_{it} + e_{it}, \quad i=1, \ldots, n,
$$

$$
t=1, \ldots, T,
$$

where $DQ_t = \sum_{i=1}^n \bar{w}_{it} DQ_{it}$ is the finite-change Divisia volume index and
\( \varepsilon_{it} \) is a zero-mean disturbance. It is to be noted that only the own (Divisia) deflated price appears in each equation; and that (14) is linear in the parameters. Both are highly attractive properties of the model.

Summing both sides of (14) over \( i=1, \ldots, n \), we get \( \sum_{i=1}^{n} \varepsilon_{it} = 0 \), which indicates that one of the \( n \) equations can be deleted [see Barten (1969)]; we delete the \( n \)th. We define the \((n-1)T\)-vector 
\[
y = [f_1' \ f_2' \ldots \ f_{n-1}']',
\]
with \( y_k \) as the \( k \)th element, where 
\[
f_i = [\tilde{w}_{i1}(Dq_{i1} - DQ_1) \ldots \tilde{w}_{iT}(Dq_{iT} - DQ_T)]' \text{ is a T-vector; the (n-1)T-}
\]
vector \( x_{1i} \) which has \( DQ_1, \ldots, DQ_T \) as elements \((T+1)\) to \( T(n+i)\) and 
zeros elsewhere, with \( x_{1ik} \) the \( k \)th element of \( x_{1i} \); and the \((n-1)T\)-
vector \( x_2 = [g_1' g_2' \ldots g_{n-1}']', \) with \( k \)th element \( x_{2k} \), where 
\[
g_i = [\tilde{w}_{i1}Dp_{i1} \ldots \tilde{w}_{iT}Dp_{iT}]' \text{ is a T-vector. We can then write (14) as}
\]
\[
y_k = \sum_{j=1}^{n-1} \beta_j x_{1jk} + \varepsilon_k, \quad k=1, \ldots, (n-1)T.
\]
Thus as Keller (1984) proposed, (15) can be estimated by OLS by regressing \( y_k \) on the \( n \) explanatory variables \( x_{11k}, \ldots, x_{1n-1k}, x_{2k} \). Although this is a simple procedure, the estimates are not efficient as the disturbances are non-spherical. We thus need to develop the GLS estimator.

We write (14) for \( i=1, \ldots, n-1 \) as
\[
y_t = X_t \delta + \varepsilon_t, \quad t=1, \ldots, T,
\]
where \( y_t = [\tilde{w}_{it}(Dq_{it} - DQ_t)] \) is an \((n-1)\)-vector; \( X_t \) is a matrix of 
order \((n-1)xn\) of which the \( i \)th row is made up of only two non-zero 
elements, \( DQ_t \) in the \( i \)th column and \( \tilde{w}_{it}Dp_{it}^* \) in the \( n \)th column; 
\( \delta = [\beta_1 \ldots \beta_{n-1} \kappa]'; \) and \( \varepsilon_t = [\varepsilon_{1t} \ldots \varepsilon_{n-1t}]' \).
A simple and plausible covariance structure is

\[ \text{var} \varepsilon_t = \lambda^2 \Omega_t, \quad \Omega_t = \bar{w}_t^* - \bar{w}_t^* \bar{w}_t^* \] \tag{17}

where \( \lambda^2 \) is a constant, \( \bar{w}_t^* = \text{diag}[\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{n-1}, t] \) and \( \bar{w}_t^* = [\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{n-1}, t]' \). It is to be noted that under (17), \( \text{var} \varepsilon_t \) is proportional to the \((n-1) \times (n-1)\) submatrix of the Slutsky matrix \( \pi \).

This result is implied by the theory of rational random behaviour [see Theil (1975/76, 1980)]. Under (17), the GLS estimator of \( \delta \)

\[ \hat{\delta} = \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} x_t \right)^{-1} \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} y_t \right) \] \tag{18}

and its variance is

\[ \text{var} \hat{\delta} = \lambda^2 \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} x_t \right)^{-1}. \] \tag{19}

The parameter \( \lambda^2 \) can be estimated consistently by

\[ \hat{\lambda}^2 = \frac{1}{(n-1)T} \sum_{t=1}^{T} \varepsilon_t' \Omega_t^{-1} \varepsilon_t, \] \tag{20}

where \( \varepsilon_t = y_t - X_t \hat{\delta} \) is the \((n-1)\)-vector of GLS residuals.

Consider the special case when each budget share in \( \Omega_t \) is replaced by its sample mean, so that this covariance matrix becomes a constant.

The scalar forms of (18)-(19) are then

\[ \hat{\kappa} = \frac{(Z_0 Z_{QQ} - \mathbf{B})/A Z_{QQ}}{Z_{QQ}}, \quad \hat{\beta}_i = \frac{(Z_{y_i Q} - \hat{\kappa} Z_{p_i Q})/Z_{QQ}}{Z_{QQ}}, \quad i = 1, \ldots, n-1, \] \tag{21}

and

\[ \text{var} \hat{\kappa} = \frac{\hat{\lambda}^2}{A}, \quad \text{var} \hat{\beta}_i = \frac{\hat{\lambda}^2 (A Z_{QQ} \bar{w}_i (1 - \bar{w}_i) + Z_{p_i Q}^2)/A Z_{QQ}^2}{Z_{QQ}}, \quad i = 1, \ldots, n-1, \] \tag{22}
where \( Z_{QQ} = \sum_{t=1}^{T} \bar{D}_{t} \), \( Z_{p_{1}Q} = \sum_{t=1}^{T} \bar{X}_{it} D_{p_{1}it} \bar{D}_{t} \), \( Z_{Q} = \sum_{t=1}^{T} \sum_{j=1}^{n} \bar{X}_{jt} D_{p_{j}jt} y_{jt} / \bar{w}_{j} \),
\[ y_{jt} = \bar{X}_{it} (D_{p_{1}it} - D_{t}), \quad A = \sum_{i=1}^{n} \left( \sum_{t=1}^{T} (\bar{X}_{it} D_{p_{1}it})^{2} - \left( \sum_{p_{1}Q} Z_{p_{1}Q} / Z_{Q} \right) \right) / \bar{w}_{i} \],
\[ Z_{ij} = \sum_{t=1}^{T} \bar{X}_{jt} D_{ij} \], \( B = \sum_{j=1}^{n} Z_{p_{j}Q}^{2} y_{j} / \bar{w}_{j} \) and \( \bar{w}_{i} \) is the sample mean of \( \bar{w}_{it} \). For the derivation of (21) and (22), see the Appendix.

As an illustrative application, we use the annual Dutch and British data from Theil (1975/76, Tables 5.2 and 5.3) to estimate the model by GLS under (17). The Dutch data cover the period 1922 to 1963 with war years excluded (\( T=31 \)) and the British 1900 to 1938 with war years excluded (\( T=30 \)). There are the same \( n=4 \) goods in both data sets. Following Theil (1975/76), for the U.K. we added a constant term to each equation to allow for trend-like changes in tastes etc. The estimates are (standard errors in parantheses):

<table>
<thead>
<tr>
<th></th>
<th>Netherlands</th>
<th>U.K.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constant</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Food</td>
<td>-</td>
<td>0.028 (.069)</td>
</tr>
<tr>
<td>Beverages</td>
<td>-</td>
<td>-0.226 (.047)</td>
</tr>
<tr>
<td>Durables</td>
<td>-</td>
<td>-0.001 (.057)</td>
</tr>
<tr>
<td>Remainder</td>
<td>-</td>
<td>0.199 (.070)</td>
</tr>
<tr>
<td>( \beta_{1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Food</td>
<td>-0.112 (.027)</td>
<td>-0.154 (.042)</td>
</tr>
<tr>
<td>Beverages</td>
<td>-0.012 (.018)</td>
<td>0.057 (.028)</td>
</tr>
<tr>
<td>Durables</td>
<td>0.222 (.026)</td>
<td>0.102 (.036)</td>
</tr>
<tr>
<td>Remainder</td>
<td>-0.098 (.029)</td>
<td>-0.004 (.043)</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.296 (.078)</td>
<td>0.560 (.062)</td>
</tr>
</tbody>
</table>

The estimates of \( \sigma \) are close to the unweighted averages of the off-diagonal elasticities of substitution at means of 0.324 and 0.581, respectively, obtained from the symmetry-constrained estimates given in
Theil (1975/76, Tables 5.6 and 5.7). When we specify the covariance matrix as a constant, the estimation results remained almost unchanged; for full details, see the Appendix.

REFERENCES


APPENDIX

A Result on Partitioned Inversion

Proposition:

Let $D$ be a non-singular matrix of order $m$, and consider

$$X = D + \alpha uu' + \beta vv',$$

where $\alpha$ and $\beta$ are scalars and $u$ and $v$ are two $m$-element vectors. Then

$$X^{-1} = D^{-1} + D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v),'$$

where $a_{11} = -\alpha(1 + \beta\mu)/r, \quad a_{21} = \alpha\beta\delta/r = a_{12}, \quad a_{22} = -\beta(1 + \alpha\gamma)/r, \quad r = (1 + \alpha\gamma)(1 + \beta\mu) - \alpha\beta\delta^2, \quad \gamma = u'D^{-1}u, \quad \delta = u'D^{-1}v$ and $\mu = v'D^{-1}v$.

provided $r \neq 0$.

Proof: We shall verify that $XX^{-1} = I$. We have

$$XX^{-1} = (D + \alpha uu' + \beta vv')(D^{-1} + D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)')$$

$$= I + u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + v(a_{21}D^{-1}u + a_{22}D^{-1}v)' + \alpha uu'D^{-1}$$

$$+ \alpha uu'D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + \alpha uu'D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)' + \beta vv'D^{-1}$$

$$+ \beta vv'D^{-1}u(a_{11}D^{-1}u + a_{12}D^{-1}v)' + \beta vv'D^{-1}v(a_{21}D^{-1}u + a_{22}D^{-1}v)',$$

$$= I + (a_{11} + \alpha + \alpha\gamma a_{11} + \alpha\delta a_{21})uu'D^{-1} + (a_{12} + \alpha\gamma a_{12} + \alpha\delta a_{22})uv'D^{-1}$$

$$+ (a_{21} + \beta\delta a_{11} + \beta\mu a_{21})vu'D^{-1} + (a_{22} + \beta + \beta\delta a_{12} + \beta\mu a_{22})vv'D^{-1}$$

$$= I + k_1 uu'D^{-1} + k_2 uv'D^{-1} + k_3 vu'D^{-1} + k_4 vv'D^{-1},$$
where

\[ k_1 = a_{11} + \alpha + \alpha \gamma a_{11} + \alpha \delta a_{21}, \]

\[ k_2 = a_{12} + \alpha \gamma a_{12} + \alpha \delta a_{22}, \]

\[ k_3 = a_{21} + \beta \delta a_{11} + \beta \mu a_{21} \]

and

\[ k_4 = a_{22} + \beta + \beta \delta a_{12} + \beta \mu a_{22}. \]

Now we will verify that \( k_1 = k_2 = k_3 = k_4 = 0. \)

Consider

\[ k_1 = \alpha + a_{11}(1 + \alpha \gamma) + \alpha \delta a_{21} \]
\[ = \alpha - \alpha(1 + \beta \mu)(1 + \alpha \gamma)/r + \alpha \delta \alpha \beta \delta /r \]
\[ = \alpha - \alpha[(1 + \alpha \gamma)(1 + \beta \mu) - \alpha \beta \delta^2]/r \]
\[ = \alpha - \alpha \]
\[ = 0, \]

\[ k_2 = a_{12} + \alpha \gamma a_{12} + \alpha \delta a_{22} \]
\[ = \alpha \beta \delta (1 + \alpha \gamma)/r - \alpha \delta \beta (1 + \alpha \gamma)/r \]
\[ = 0, \]

\[ k_3 = a_{21}(1 + \beta \mu) + \beta \delta a_{11} \]
\[ = \alpha \beta \delta (1 + \beta \mu)/r - \beta \delta \alpha (1 + \beta \mu)/r \]
\[ = 0, \]
\[ k_4 = \beta + (1 + \beta \mu) a_{22} + \beta \delta a_{12} \]
\[ = \beta - \beta (1 + \beta \mu)(1 + \alpha \gamma)/r + \beta \delta \alpha \beta \delta /r \]
\[ = \beta - \beta [(1 + \alpha \gamma)(1 + \beta \mu) - \alpha \beta \delta^2]/r \]
\[ = \beta - \beta \]
\[ = 0. \]

Therefore \( XX^{-1} = I \). This completes the proof.

Derivation of Equation (13)

From equation (12), we have

\[ v = K W - K \omega \omega' + \phi \theta \theta' \]

where \( W \) is a non-singular diagonal matrix of order \( n \) and \( \omega, \theta \) are \( n \)-vectors. Thus using the above proposition,

\[ v^{-1} = \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} W^{-1} \omega (a_{11} W^{-1} \omega + a_{12} W^{-1} \theta)' + \frac{1}{\kappa^2} W^{-1} \theta (a_{21} W^{-1} \omega + a_{22} W^{-1} \theta)' \]
\[ = \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} \omega (a_{11} \eta + a_{12} \eta)' + \frac{1}{\kappa^2} \theta (a_{21} \eta + a_{22} \eta)' \]
\[ = \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} a_{11} \eta \eta' + \frac{1}{\kappa^2} a_{12} (\eta' + \eta''') + \frac{1}{\kappa^2} a_{22} \eta \eta' \]

where \( \eta = [\eta_1 \eta_2 \ldots \eta_n]' \), \( a_{11} = \kappa (1 + \frac{1}{\kappa} \phi \omega W^{-1} \omega)'/r \]
\[ = \kappa + \phi \sum_{i=1}^{n} (1, \eta_i)' /r \]
\[ a_{12} = \phi \omega W^{-1} \theta'/r = \phi'/r = a_{21}, a_{22} = \phi (1 - \omega W^{-1} \omega)/r \]
\[ = 0 \text{ and } r = (1 - w'W^{-1} w)(1 + \phi W^{-1} \theta) + \kappa \phi (w W^{-1} \theta/k)^2 = \phi/k \neq 0. \]

Thus we have

\[ v^{-1} = \frac{1}{\kappa} W^{-1} + \frac{1}{\kappa^2} a_{11} \eta \eta' - \frac{1}{\kappa} (\eta' + \eta'''). \]
Therefore, if we denote $v^{-1} = [v_{ij}]$, we have

$$v_{ij} = \frac{1}{\kappa} a_{11} + \frac{1}{\kappa w_i} \delta_{ij} - \frac{1}{\kappa} (\eta_i + \eta_j).$$

From equation (2), we can write this as

$$\frac{M}{\lambda} \frac{1}{p_i p_j} \frac{\delta^2 u}{\delta q_i \delta q_j} = \frac{1}{\kappa} a_{11} + \frac{1}{\kappa w_i} \delta_{ij} - \frac{1}{\kappa} (\eta_i + \eta_j).$$

Therefore,

$$\frac{\delta^2 u}{\delta (p_i q_i) \delta (p_j q_j)} = \alpha + \frac{\beta}{w_i} \delta_{ij} - \beta (\eta_i + \eta_j),$$

where $\alpha = (\lambda/M\kappa^2) a_{11} ; \quad \alpha = \kappa (\kappa + \phi \sum_i \theta_i \eta_i )/\phi$; and $\beta = \lambda/M\kappa$. The above is equation (13). Consider the case $i \neq j$,

$$\frac{\delta^2 u}{\delta (p_i q_i) \delta (p_j q_j)} = \alpha - \beta (\eta_i + \eta_j).$$

Therefore

$$\frac{\delta^2 u}{\delta (p_i q_i) \delta (p_j q_j)} > 0 \quad \text{iff} \quad \alpha > \beta (\eta_i + \eta_j)$$

$$\text{iff} \quad \frac{\alpha}{\beta} > (\eta_i + \eta_j), \quad \text{as} \quad \beta < 0$$

$$\text{iff} \quad \kappa + \phi \sum_i \theta_i \eta_i > \phi (\eta_i + \eta_j), \quad \text{as} \quad \phi < 0$$

$$\text{iff} \quad \kappa > \phi [- \sum_i \theta_i \eta_i + \eta_i \eta_j]$$

$$\text{iff} \quad \tau < \sum_i \theta_i \eta_i,$$

where $\tau = \kappa/\phi$ and

$$\theta^*_i = \begin{cases} -\theta_k & \text{if } k \neq i, j \\ (1 - \theta_k) & \text{if } k = i, j \end{cases}$$
so that $\sum_{k} \theta_k^* = 1$.

**Derivation of (21)-(22)**

From equations (18)-(19) we have

$$\hat{\delta} = \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} x_t \right)^{-1} \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} y_t \right), \quad \text{var} \hat{\delta} = \lambda^2 \left( \sum_{t=1}^{T} x_t' \Omega_t^{-1} x_t \right)^{-1}.$$

We replace each budget share in $\Omega_t$ with its sample mean,

(A1) \quad $\Omega = \bar{\Omega} - \bar{w} \bar{w}'$,

where $\bar{\Omega} = \text{diag}(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{n-1})$ and $\bar{w} = [\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{n-1}]$. If we take the inverse of both sides of (A1), we get

$$\Omega^{-1} = \bar{\Omega}^{-1} + \frac{1}{\bar{w}_n} \bar{w} \bar{w}' .$$

Now we consider

$$x_t' \Omega_t^{-1} x_t = \begin{pmatrix} DQ_t & I_{n-1} \\ * & * \\ p_t & * \end{pmatrix} \Omega^{-1} \begin{pmatrix} DQ_t & I_{n-1} \\ * & * \\ p_t & * \end{pmatrix}^*$$

$$= \begin{pmatrix} DQ_t & \bar{\Omega}^{-1} \\ p_t' & \bar{\Omega}^{-1} \end{pmatrix} \begin{pmatrix} DQ_t & I_{n-1} \\ * & * \\ p_t & * \end{pmatrix}^*$$

$$= \begin{pmatrix} DQ_t & DQ_t \bar{\Omega}^{-1} p_t^* \\ DQ_t p_t' & p_t' \bar{\Omega}^{-1} p_t^* \end{pmatrix}.$$
where \( \tilde{p}_t^* = [\tilde{w}_{it} D_{p_t}^*] \) is a \((n-1)\)-vector. Therefore

\[
(A2) \quad \sum_{t=1}^{T} x_t' \Omega^{-1} x_t = \begin{bmatrix}
Z_{QQ} \Omega^{-1} & \Omega^{-1} z_{pQ} \\
z_{pQ}' \Omega^{-1} & p
\end{bmatrix},
\]

where \( Z_{QQ} = \sum_{t=1}^{T} D_{Q_t}^2 ; \quad z_{pQ} = [Z_{p_1 Q} Z_{p_2 Q} \cdots Z_{p_{n-1} Q}]' \), \( Z_{p_1 Q} = \sum_{t=1}^{T} \tilde{w}_{it} D_{p_t}^* D_{Q_t} \)

and \( p^* = \sum_{t=1}^{T} p_t^* \Omega^{-1} p_t^* = \sum_{t=1}^{T} (\tilde{w}_{it} D_{p_t}^*)^2 / \tilde{w}_i \).

Taking the inverse of both sides of (A2) gives

\[
(A3) \quad \left\{ \sum_{t=1}^{T} x_t' \Omega^{-1} x_t \right\}^{-1} = \begin{bmatrix}
\Omega / Z_{QQ} + z_{pQ} z_{pQ}' / Z_{QQ}^2 & - z_{pQ} / AZ_{QQ} \\
- z_{pQ}' / AZ_{QQ} & 1 / A
\end{bmatrix},
\]

where \( A = p^* - (z_{pQ}' \Omega^{-1} z_{pQ}) / Z_{QQ} = \sum_{t=1}^{T} \sum_{t=1}^{T} (\tilde{w}_{it} D_{p_t}^*)^2 - z_{pQ}' / Z_{QQ} / \tilde{w}_i \).

Similarly,

\[
X_t' \Omega^{-1} y_t = \begin{bmatrix}
D_{Q_t} Q_t^{-1} y_t \\
p_t^* \Omega^{-1} y_t
\end{bmatrix}
\]

which gives

\[
(A4) \quad \sum_{t=1}^{T} x_t' \Omega^{-1} y_t = \begin{bmatrix}
\Omega^{-1} z_{yQ} \\
\sum_{t=1}^{T} p_t^* \Omega^{-1} y_t
\end{bmatrix},
\]

where \( z_{yQ} = [Z_{y_1 Q} \cdots Z_{y_{n-1} Q}]' \), \( Z_{y_1 Q} = \sum_{t=1}^{T} \tilde{w}_{it} D_{Q_t} \) and \( y_t = \tilde{w}_{it} (D_{Q_t} - D_{Q_t}^*) \).
Thus from (A3) and (A4) we can write

$$\hat{\delta} = \left( \begin{array}{c} \Omega/z_{QQ} + z_{pQ}z_{pQ}^\top/\Omega z_{QQ}^2 - z_{pQ}/AZ_{QQ} \\ -z_{pQ}/AZ_{QQ} \end{array} \right) \left( \begin{array}{c} \Omega^{-1}z_{yQ} \\ \frac{T}{t=1} p_{t}^\top \Omega^{-1}y_{t} \end{array} \right).$$

As $$\hat{\delta} = [\hat{\beta} \hat{\kappa}]^\top$$, we get

$$\hat{\kappa} = -z_{pQ}^\top\Omega^{-1}z_{yQ}/AZ_{QQ} + \frac{T}{t=1} p_{t}^\top\Omega^{-1}y_{t}/A$$

$$= -B/AZ_{QQ} + z_{0}/A$$

(A5)  

$$= (z_{0}z_{QQ} - B)/AZ_{QQ},$$

where

$$z_{0} = \sum_{j=1}^{n}\frac{p_{j}^\top\Omega^{-1}y_{j}}{\tilde{w}_{j}} = \sum_{j=1}^{n} \left( \sum_{t=1}^{T} \tilde{w}_{j} p_{t}^\top y_{t}/\tilde{w}_{j} \right), \quad B = z_{pQ}^\top\Omega^{-1}z_{yQ}$$

$$= \sum_{j=1}^{n} z_{pQ}^\top y_{j} Q_{j}/\tilde{w}_{j};$$

and

$$\hat{\beta} = (\Omega/z_{QQ} + z_{pQ}z_{pQ}^\top/\Omega z_{QQ}^2 ) (\Omega^{-1}z_{yQ}) - z_{pQ} \left( \sum_{t=1}^{T} p_{t}^\top\Omega^{-1}y_{t} \right)/AZ_{QQ}$$

$$= z_{yQ}/z_{QQ} + z_{pQ}z_{pQ}^\top\Omega^{-1}z_{yQ}/AZ_{QQ}^2 - z_{pQ} \left( \sum_{t=1}^{T} p_{t}^\top\Omega^{-1}y_{t} \right)/AZ_{QQ}$$

$$= z_{yQ}/z_{QQ} + Bz_{pQ}/AZ_{QQ}^2 - z_{0}z_{pQ}/AZ_{QQ}$$

$$= \left( z_{yQ} - (z_{0}z_{QQ} - B)z_{pQ} \right)/AZ_{QQ}$$

$$= (z_{yQ} - \hat{\kappa}z_{pQ})/z_{QQ}.$$  

Thus we have

(A6)  

$$\hat{\beta}_{i} = (z_{y_{i}Q} - \hat{\kappa}z_{p_{i}Q})/z_{QQ} \quad \text{for } i=1,\ldots,n-1.$$  

Equations (A5) and (A6) are identical to equation (21) of the text.
Using (A3) we have

\[ \text{var } \hat{\delta} = \lambda^2 \left( \frac{\Omega / Z_{QQ} + Z_{pQ}Z_{pQ}' / AZ_{QQ}^2}{Z_{pQ}' / AZ_{QQ}} - \frac{Z_{pQ} / AZ_{QQ}}{1/A} \right). \]

Thus

\[ \text{var } \hat{\kappa} = \lambda^2 / A \]

and

\[ \text{var } \hat{\beta} = \lambda^2 (\Omega / Z_{QQ} + Z_{pQ}Z_{pQ}' / AZ_{QQ}^2) \]

or equivalently,

\[ \text{var } \hat{\beta}_i = \lambda^2 (AZ_{QQ} \bar{w}_i (1 - \bar{w}_i) + Z_{pQ}^2 / AZ_{QQ}^2), \quad i = 1, \ldots, n-1. \]

Equations (A8) and (A9) are equivalent to (22).

**The Estimation Results**

Below we present the estimates of the model with standard errors given in parentheses and where \( \alpha_i \) is the constant term in equation \( i \):

**Under Specification (17)**

<table>
<thead>
<tr>
<th></th>
<th>Netherlands (without constants)</th>
<th>U.K. (with constants)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta_1 = -0.112 (.027) )</td>
<td>( \beta_1 = -0.154 (.042) )</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>( = -0.012 (.018) )</td>
<td>( \beta_2 = 0.057 (.028) )</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>( = 0.222 (.026) )</td>
<td>( \beta_3 = 0.102 (.036) )</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>( = -0.098 (.029) )</td>
<td>( \beta_4 = -0.004 (.043) )</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>( = -0.296 (.078) )</td>
<td>( \kappa = -0.560 (.062) )</td>
</tr>
</tbody>
</table>
Under Specification of Constant Covariance Matrix

\[ \beta_1 = -1.11 (0.027) \quad \beta_1 = -1.151 (0.042) \quad \kappa_1 = 0.32 (0.068) \]

\[ \beta_2 = -0.12 (0.017) \quad \beta_2 = 0.54 (0.028) \quad \kappa_2 = -2.25 (0.046) \]

\[ \beta_3 = 0.222 (0.026) \quad \beta_3 = 0.104 (0.036) \quad \kappa_3 = 0.09 (0.058) \]

\[ \beta_4 = -0.099 (0.029) \quad \beta_4 = -0.007 (0.043) \quad \kappa_4 = 0.201 (0.070) \]